Chapter 2

2 Fundamentals of pushing object

Most objects are at rest most of the time. To move an object a force is needed to be applied on it. There are two frictional contacts needed to be addressed when moving an object. One is the friction between the terrain and the object. The other is the friction between the moving object and the pusher.

2.1 Coulomb's Law

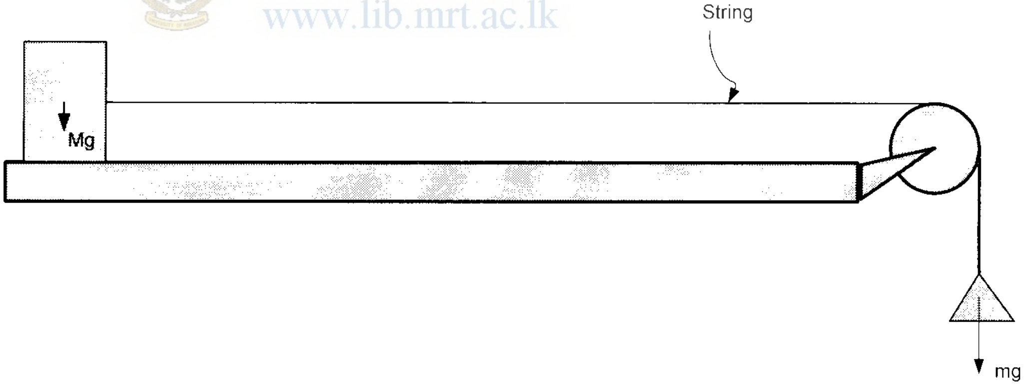

Figure 2-1: Test for friction force

Consider a block sliding on a horizontal surface, pulled by a string as shown in figure 2-1. Assume the two surfaces are fairly clean, dry, and unlubricated. Imagine that the instruments are available to measure the force f_a applied via the string, and the tangential force f_t due to friction between the table and the block. If the applied force

f_a increased gradually, the behavior illustrated in figure 2-2 can be observed. For small applied forces, the frictional force will balance the applied force, so that the block does not move. Above some threshold, the block will begin moving, and the frictional force will now be constant. If a large number of experiments are carried out by varying the block's mass, the block's shape, the materials, and so forth, the factors for the limiting frictional forces can be identified as the normal force, and the materials involved. If f_{ts} is the limiting value of static friction, at which motion begins, and f_{td} is the value of dynamic friction, then these values are approximately:

$$f_{ts} = \mu_s f_n$$

$$f_{td} = \mu_d f_n$$

Where f_n is the normal force, μ_s is the static coefficient of friction, and μ_d is the dynamic coefficient of friction. Typically μ_s is larger than μ_d, but this difference is ignored and speak of a single coefficient of friction μ . A simple statement of Coulomb's law is:

$$|\boldsymbol{f_t}| \leq \mu|\boldsymbol{f_n}|$$

if there is no motion, and if there is any motion, with the tangential force in a direction opposite to the motion.

$$|\boldsymbol{f_t}| = \mu|\boldsymbol{f_n}|$$

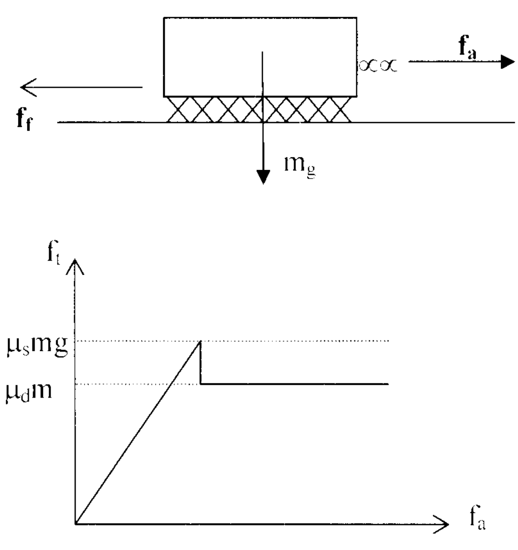

Figure 2-2 Coulomb's law of sliding friction

In particular, the tangential force is largely independent of the contact area, and of the velocity of motion. The coefficient of friction is considered to be a material property, depending only on the materials involved. Tables of the coefficient of friction should not be taken too seriously, but some typical values are given below:

Table 1: The friction coefficient between some materials

Materials	μ
metal on metal	0.15-0.6
rubber on concrete	0.6-0.9
plastic wrap- on lettuce	∞
Leonardo's number	0.25

Experiments like those described above provide the basis of Coulomb's law, which will be stated more carefully later. The history of the law is also interesting. Coulomb's work with friction was his first scientific achievement. Coulomb's interest in friction was spurred by practical engineering matters. He was a military engineer in carrier and carried his huge laboratory apparatus with him from one assignment to another. Coulomb is also known for the invention of the torsion balance and for his studies of electricity, leading to the law for electrostatic attraction which is unfortunately also known as Coulomb's law.

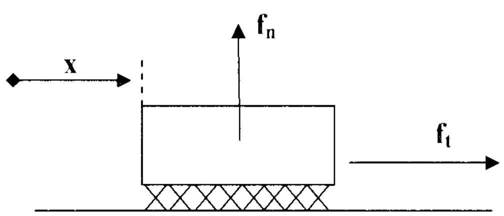

Figure 2-3 Sliding block with Coulomb friction

Coulomb was not the first to propose Coulomb's law of sliding friction. Amontons had proposed it earlier, and the law is occasionally referred to as Amontons's law. It also appears that Leonardo da Vinci had earlier posed a more restrictive version of the law, supposing the coefficient of friction to be always one fourth. Coulomb's law is a phenomenological law, providing an approximate description of an aggregate behavior. For that reason there are some who would prefer not to call it a law at all. There are more fundamental approaches to the modeling of friction, and more precise approximations of friction. But Coulomb's law still provides the best combination of simplicity and accuracy for many purposes.

2.2 Single degree-of-freedom problems

Lets begin by considering the simplest problems, involving just one degree of freedom. Consider a block in frictional contact with a supporting plane, which is prevented somehow from moving away from the plane (fig. 2-3). The tangential position of the block is given by x, and the frictional force is given by f_n and f_t, respectively normal and tangential to the supporting plane. Coulomb's law prescribes a constraint on the contact force, depending on the *contact mode* indicated in the table2

Table 2: Contact force and contact mode

$\dot{x}$	$\ddot{x}$		
< 0		$f_t = \mu f_n$	left sliding
> 0		$f_t = -\mu f_n$	right sliding
$= 0$	<0	$f_t = \mu f_n$	left sliding
$= 0$	>0	$f_t = -\mu f_n$	right sliding
$= 0$	$=0$	$\lvert f_t \rvert \le \mu f_n$	rest

Now suppose that a gravitational field is introduced. So that the supporting surface is an inclined plane (fig. 2-4). Let α be the angle of the inclined plane with respect to the horizontal. What is the maximum angle α at which the block can remain at rest?

If the block is at rest, then the gravitational force must balance the total contact force:

$$f_n = mg\ cos\alpha$$

$$f_t = mg\ sin\alpha$$

At rest we have $|f_t| \leq |\mu f_n|$. The limiting case is given by

$$f_t = \mu f_n$$

Substituting,

$$mg\ \sin\alpha = \mu mg\ \cos\alpha$$

$$\alpha = \tan^{-1}\mu$$

The desired angle α is the arc-tangent to the coefficient of friction. This angle is sometimes called the friction angle or the angle of repose.

The friction angle provides an elegant geometrical approach to Coulomb's law. Consider all the forces satisfying Coulomb's law for an object at rest, i.e. all the forces satisfying the condition

$$|f_t| \leq |\mu f_n|$$

This set of forces describes a cone in the force space, called the *friction cone*, with vertex at the origin, and dihedral angle $2tan^{-1}\mu$ (fig.2-4). Then Coulomb's law can be stated as

For left sliding $\quad f_n + f_t \in$ right edge of friction cone

For right sliding $f_n + f_t \in$ left edge of friction cone

For rest $f_n + f_t \in$ friction cone

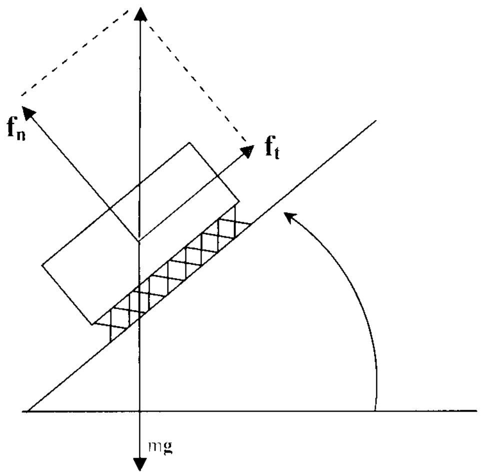

Figure 2-4 Sliding block on inclined plane

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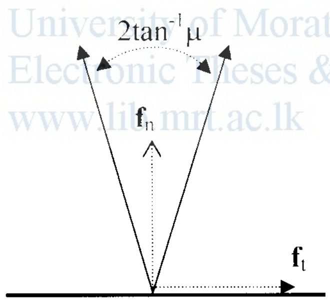

Figure 2-5 The friction cone

To satisfy Coulomb's law the contact force deviates by at most $\tan^{-1}\mu$ from the contact normal.

2.3 Planar sliding

Some manipulation tasks involve an object sliding on a planar support surface. The motion of a pushed object is often undefined. If a rigid object is supported by more than three contact points, the distribution of supporting forces are underdetermined. If the frictional forces are assumed proportional to the normal forces, as Coulomb suggests, then the frictional forces are also underdetermined. The problem is illustrated by the defective dinner plate of fig.2-6. The plate was designed with a circular ridge on the bottom, so that the supporting forces would be concentrated at the edge of the plate. Unfortunately, the bottom of the plate sagged during the firing process, so that the center is also in contact with the planar support. There is no way to predict whether the support forces will be concentrated at the center, giving it an irritating tendency to rotate, or at the edge, resisting rotation. In practice, the plate's behavior will depend on details that may be very difficult to model. It may behave well with a tablecloth, and poorly without a tablecloth. Its behavior might depend on the phase of the moon. (Tidal forces induce microscopic changes in the shapes of the plate and table.)

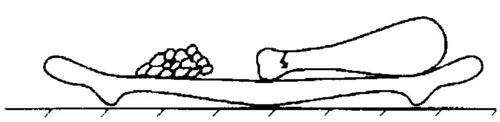

Figure 2-6 Cross section of a plate

It is impossible to predict the motion of this plate, without knowing the distribution of supporting forces between the plate and the table.

2.4 Force and moment of planar sliding

Let some object be in planar motion, supported by a fixed planar surface. Choose a coordinate frame with the *x-y* plane coincident with the support, and *z* pointing outward. Let the object's contact with the surface be confined to some region *R*. Let **r** be the position vector of some point in the object, and let *v(r)* be the velocity of that point.

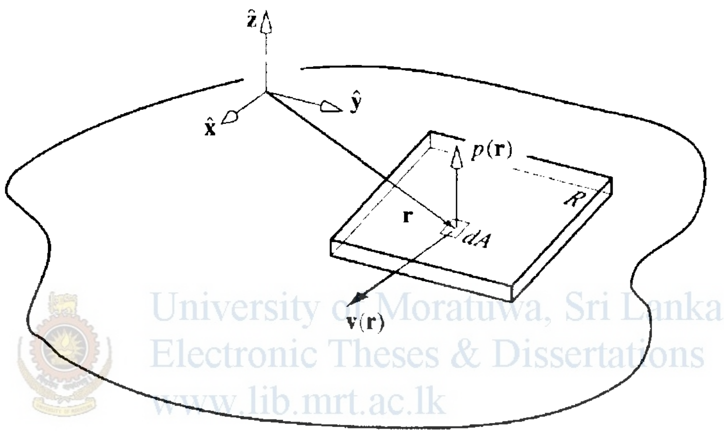

Figure 2-7 Notation for planar sliding.

If *p(r)* is the pressure at **r**, and *dA* a differential element of area at **r**, then the magnitude of the normal force at **r** is given by

$$p(\boldsymbol{r})dA$$

and Coulomb's law gives us the tangential force at **r**:

$$-\mu\frac{\boldsymbol{v}(\boldsymbol{r})}{|\boldsymbol{v}(\boldsymbol{r})|}p(\boldsymbol{r})dA$$

for $|\mathbf{v(r)}| \neq 0$, where μ is the coefficient of friction, assumed uniform over the contact region R.

Integrating over R, we obtain expressions for the total force and moment due to friction:

$$\boldsymbol{f}_f = -\mu \int_R \frac{\boldsymbol{v}(\boldsymbol{r})}{|\boldsymbol{v}(\boldsymbol{r})|} p(\boldsymbol{r}) dA \quad \rightarrow 2.1$$

$$\boldsymbol{n}_f = -\mu \int_R \boldsymbol{r} \times \frac{\boldsymbol{v}(\boldsymbol{r})}{|\boldsymbol{v}(\boldsymbol{r})|} p(\boldsymbol{r}) dA \quad \rightarrow 2.2$$

Note that the frictional force $\mathbf{f}_f$ lies in the x-y plane; and the total frictional moment $\mathbf{n}_f$ acts along the z-axis. Without knowledge of the pressure distribution $p(r)$, these integrals cannot be evaluated, leading to indeterminacy in the frictional forces. There is an exception, though: pure translation.

2.4.1 Case 1: Pure translation

If the object is in pure translation, all points are moving in the same direction, and we can factor the integrals of equations 2.1 and 2.2.

$$\boldsymbol{f}_f = -\mu \frac{\boldsymbol{v}(\boldsymbol{r})}{|\boldsymbol{v}(\boldsymbol{r})|} \int_R p(\boldsymbol{r}) dA \quad \rightarrow 2.3$$

$$\boldsymbol{n}_f = -\mu \int_R \boldsymbol{r} p(\boldsymbol{r}) dA \times \frac{\boldsymbol{v}(\boldsymbol{r})}{|\boldsymbol{v}(\boldsymbol{r})|} \quad \rightarrow 2.4$$

Let $\mathbf{f}_0$ be the total normal force, and let $\mathbf{r}_0$ be the centroid of the pressure distribution.

Then

$$f_o = \int_R p(\boldsymbol{r})dA$$

$$\boldsymbol{r}_0 = \frac{1}{f_0}\int_R \boldsymbol{r}p(\boldsymbol{r})dA$$

Substituting into equations 2.3 and 2.4,

$$\boldsymbol{f}_f = -\mu \frac{\boldsymbol{v}(\boldsymbol{r})}{|\boldsymbol{v}(\boldsymbol{r})|} f_0$$

$$\boldsymbol{n}_f = \boldsymbol{r}_0 \times \boldsymbol{f}_f$$

Hence the frictional forces distributed over the supporting region have a resultant, with magnitude μf_0, in a direction opposing the motion, through the centroid $\mathbf{r}_0$. In other words, the force is equivalent to that obtained by applying Coulomb's law to the sliding of a single point located at $\mathbf{r}_0$.

(The center of friction (COF) is the centroid $\mathbf{r}_0$ of the pressure distribution.)

In some cases, the center of friction is easily determined. If an object is at rest on the supporting plane, with no applied forces other than gravity and the supporting contact forces, then the center of friction is directly below the center of gravity. This is the only location that allows the contact forces to balance the gravitational force. We can generalize slightly, allowing additional applied forces, as long as they are in the supporting plane. We can also permit accelerated motion of the body, if the center of gravity is in the supporting plane. But acceleration of a body whose center of gravity is above the supporting plane it will in general, cause a shift in the pressure distribution, and a corresponding shift in the center of friction. Applied forces not lying in the supporting plane will generally cause a similar shift.

2.4.2 Case 2: Rotation

Now suppose that the body is rotating, with an instantaneous center r_c. Then the velocity of a point at r is given by

$$v(r) = \omega \times (r - r_c)$$

$$= \dot{\theta}\hat{k} \times (r - r_c)$$

and the direction of motion at **r** is

$$\frac{v(r)}{|v(r)|} = sgn(\dot{\theta})\hat{k} \times \frac{(r - r_c)}{|r - r_c|}$$

Substituting into equations 2.1 and 2.2 we obtain

$$f_f = -\mu\, sgn(\dot{\theta})\hat{k} \times \int_R \frac{(r - r_c)}{|r - r_c|} p(r)dA$$

$$n_{fz} = -\mu\, sgn(\dot{\theta}) \int_R r \,.\, \frac{(r - r_c)}{|r - r_c|} p(r)dA$$

These equations have a well-defined limit as the rotation center $\mathbf{r}_c$ approaches infinity, so they apply to pure translations as well as rotations.