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BRANNAN'S CONJECTURE AND TRIGONOMETRIC SUMS

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ABSTRACT. We prove some versions of Brannan's Conjecture on Taylor coefficients of the ratio of two binomials of the form $(1+zx)^\alpha/(1-x)^\beta$ and discuss some related inequalities for trigonometric sums.

1. Brannan's conjecture

Let

$$\frac{(1+zx)^\alpha}{(1-x)^\beta} = \sum_{n=0}^{\infty} A_n(\alpha,\beta,z)x^n, \tag{1.1}$$

where $\alpha > 0$, $\beta > 0$, and $z = e^{i\theta}$, $0 \le \theta \le 2\pi$.

Is it true that

$$|A_n(\alpha,\beta,z)| \le A_n(\alpha,\beta,1) \tag{1.2}$$

for all $\alpha > 0$, $\beta > 0$, all z such that $|z| = 1$, and all odd integers n?

This question, raised by D. A. Brannan [4] in the context of the coefficient problem for functions with bounded boundary rotation, has remained open since 1973. Brannan himself verified (1.2) for $n = 3$ and also noticed that inequality (1.2) is not true for even n in general [4]. After R. W. Barnard published his survey paper [2], where a history of the problem up till 1990 can be found, the inequality (1.2) became known as *Brannan's conjecture*.

The first essential progress was achieved by D. Aharonov and S. Friedland [1] who proved (1.2) for all $\alpha \ge 1$, $\beta > 1$ and all n. Later progress can be found in [7]. In a recent paper by S. Ruscheweyh and L. Salinas [8], inequality (1.2) was proved for all odd n and all α and β such that $\alpha = \beta$.

The case $0 < \alpha \le 1$, $0 < \beta \le 1$ turned out to be rather difficult. In this paper, we deal mainly with a special case of it when $0 < \alpha < 1$ and $\beta = 1$. This case of the Brannan conjecture has drawn the attention of several researchers. For $n = 5$ and $\beta = 1$, inequality (1.2) was proved by J. G. Milcetich [6] and for $n = 7$ and $\beta = 1$, it was established by R. W. Barnard, K. Pearce, and W. Wheeler [3].

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Recently, U. C. Jayatilake developed a squaring procedure, which allowed him to give a computer-assisted proof of the inequality (1.2) for $\beta = 1$ and all odd $n \leq 51$ [5].

We find it more natural to treat Brannan's coefficients $A_n(\alpha, \beta, z)$ as analytic functions of a complex variable $z = re^{i\theta}$. Brannan's problem can then be stated in the following more general form.

Given $\alpha > 0$, $\beta > 0$ and a positive integer n, find the largest $r_n = r_n(\alpha, \beta)$ such that

$$|A_n(\alpha, \beta, z)| \leq A_n(\alpha, \beta, r) \tag{1.3}$$

for all $z = re^{i\theta}$ with $r \leq r_n(\alpha, \beta)$.

In this setting, Brannan's conjecture suggests that $r_n(\alpha, \beta) \geq 1$ if n is odd. Our first theorem shows that $r_n(\alpha, 1) \geq 1/2$ for all odd n.

Theorem 1. *Let $0 < \alpha < 1$ and let $z = re^{i\theta}$ with $0 < \theta < 2\pi$ and $0 < r \leq 1/2$. Then*

$$|A_{2n-1}(\alpha, 1, z)| < A_{2n-1}(\alpha, 1, r), \quad \textit{for all } n \in \mathbb{N}. \tag{1.4}$$

Our second theorem shows that in the case $\beta = 1$ a weaker version of inequality (1.3) holds for all z such that $|z| \leq 1$.

Theorem 2. *Let $0 < \alpha < 1$ and let $z = re^{i\theta}$ with $0 < \theta < 2\pi$ and $0 < r \leq 1$. Then*

$$\Re\, A_{2n-1}(\alpha, 1, z) < A_{2n-1}(\alpha, 1, r), \quad \textit{for all } n \in \mathbb{N}. \tag{1.5}$$

Two key innovations used in this paper are new integral representations of Brannan's coefficients and treating these coefficients as analytic functions of the complex variable. The latter allows us to estimate the radial growth of these coefficients. In the last section, we discuss briefly a relationship of our results to some classical inequalities for trigonometric sums.

2. Integral representation of Brannan's coefficients

Our main results are obtained for the case $\beta = 1$. Accordingly, in this case we will use a shorter notation $A_n(\alpha, z)$ instead of $A_n(\alpha, 1, z)$. Expanding $(1 + zx)^\alpha/(1 - x)$ into a Taylor series in z, we obtain

$$\frac{(1+zx)^\alpha}{1-x} = \sum_{n=0}^{\infty} \frac{(-\alpha)_n(-1)^n}{n!}(zx)^n \cdot \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-\alpha)_k(-1)^k z^k}{k!} x^n.$$

Here $(\alpha)_k$ is the Pochhammer symbol, defined by the equation

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)$$

when $k > 0$ and $(\alpha)_k = 1$ when $k = 0$. Hence, Brannan's coefficients can be represented as

$$A_n(\alpha, z) = \sum_{k=0}^{n} \frac{(-\alpha)_k(-z)^k}{k!}. \tag{2.1}$$

Using the well-known representation of the Pochhammer symbol as a quotient of two Euler gamma functions, we obtain:

$$\begin{aligned}(-\alpha)_k &= \frac{\Gamma(-\alpha+k)}{\Gamma(-\alpha)} = \frac{1}{\Gamma(-\alpha)}\frac{\Gamma(k)}{\Gamma(\alpha)}\frac{\Gamma(-\alpha+k)\Gamma(\alpha)}{\Gamma(-\alpha+k+\alpha)} \\ &= \frac{(k-1)!}{\Gamma(\alpha)\Gamma(-\alpha)}\int_0^1 t^{-\alpha+k-1}(1-t)^{\alpha-1}\,dt.\end{aligned}$$

Substituting this integral representation into (2.1), we can represent Brannan's coefficients in the form

$$\begin{aligned}A_n(\alpha, z) &= 1 + \sum_{k=1}^{n} \frac{(-z)^k}{k!}\frac{(k-1)!}{\Gamma(\alpha)\Gamma(-\alpha)}\int_0^1 t^{-\alpha+k-1}(1-t)^{\alpha-1}\,dt \\ &= 1 + \frac{1}{\Gamma(\alpha)\Gamma(-\alpha)}\int_0^1 \left(\sum_{k=1}^{n}\frac{(-tz)^k}{k}\right) t^{-\alpha-1}(1-t)^{\alpha-1}\,dt.\end{aligned}$$

Summarizing our observations, we obtain the following.

Lemma 1. *For real α, $0 < \alpha < 1$, and complex z, consider the Taylor series expansion*

$$\frac{(1+zx)^\alpha}{1-x} = \sum_{n=0}^{\infty} A_n(\alpha, z)x^n. \tag{2.2}$$

Then the coefficients $A_n(\alpha, z)$ of the series expansion (2.2) can be represented as

$$A_n(\alpha, z) = 1 + \frac{1}{\Gamma(\alpha)\Gamma(-\alpha)}\int_0^1 C_n(-tz)t^{-\alpha-1}(1-t)^{\alpha-1}\,dt, \tag{2.3}$$

where the function $C_n(z)$ is defined by

$$C_n(z) = \sum_{k=1}^{n} \frac{z^k}{k}. \tag{2.4}$$

Furthermore, since $A_n(\alpha, z) = A_n(\alpha, -|z|) + (A_n(\alpha, z) - A_n(\alpha, -|z|))$, the coefficients $A_n(\alpha, z)$ can be represented in the form

$$A_n(z, \alpha) = A_n(-|z|, \alpha) + \frac{1}{\Gamma(\alpha)\Gamma(-\alpha)}\int_0^1 D_n(-tz)t^{-\alpha-1}(1-t)^{\alpha-1}\,dt \tag{2.5}$$

with the function $D_n(z)$ *defined by*

$$D_n(z) = C_n(z) - C_n(|z|) = \sum_{k=1}^{n} \frac{z^k - |z|^k}{k}. \tag{2.6}$$

3. Inequalities for the real part of Brannan's coefficients

Using representation (2.3) of Brannan's coefficients $A_n(\alpha, z) = A_n(\alpha, 1, z)$ and taking into account that $\Gamma(\alpha)\Gamma(-\alpha) < 0$ for $0 < \alpha < 1$, we conclude that, in the case under consideration, inequality (1.5) holds for all $n \geq 1$ if and only if the following inequality holds for all odd $n \geq 1$:

$$\int_0^1 \Re C_n(-tz) t^{-\alpha-1}(1-t)^{\alpha-1}\, dt > \int_0^1 \Re C_n(-t|z|) t^{-\alpha-1}(1-t)^{\alpha-1}\, dt. \tag{3.1}$$

Now, for odd $n \geq 1$, inequality (3.1) follows from inequality (3.2) given in Lemma 2 below, which is of independent interest.

Lemma 2. *For any odd integer* $n \geq 1$ *and complex* z *such that* $|z| \leq 1$, *we have*

$$\Re\, C_n(-|z|) \leq \Re C_n(z), \tag{3.2}$$

with the sign of equality if and only if $z = -|z|$.

Proof. We fix odd $n \geq 1$ and consider the difference function

$$F(r, \theta) = \Re\,(C_n(z)) - C_n(-r), \quad \text{where } z = re^{i\theta}. \tag{3.3}$$

We note that $\Re C_n(-|z|) = C_n(-r)$. Therefore inequality (3.2) is equivalent to the inequality $F(r, \theta) > 0$ for all r and θ such that $0 < r \leq 1$ and $0 \leq \theta < \pi$.

We claim that for any fixed θ, $0 \leq \theta < \pi$, the function $F(r, \theta)$ considered as a function of r does not have local minima on $0 < r < 1$. To prove this claim, we will use the first and second partial derivatives $F_1(r, \theta) = \frac{\partial}{\partial r} F(r, \theta)$ and $F_{11}(r, \theta) = \frac{\partial^2}{\partial r^2} F(r, \theta)$ of the function $F(r, \theta)$ with respect to r. First, differentiating $C_n(z)$ with respect to r, we find

$$\frac{\partial}{\partial r} C_n(z) = \frac{z}{r} C_n'(z) = \frac{z}{r} \sum_{k=1}^{n} z^{k-1} = \frac{z}{r} \frac{1 - z^n}{1 - z} = \frac{z}{r} \frac{1 - \overline{z} - z^n + \overline{z} z^n}{|1 - z|^2}.$$

Taking the real part in this equation, we obtain:

$$\frac{\partial}{\partial r} \Re C_n(z) = \frac{-r + \cos\theta + r^{n+1} \cos n\theta - r^n \cos(n+1)\theta}{1 - 2r\cos\theta + r^2}. \tag{3.4}$$

For $\theta = \pi$ and any odd $n \geq 1$, Equation (3.4) can be simplified to obtain the following:

$$\frac{\partial}{\partial r} \Re C_n(-r) = -\frac{r + 1 - r^{n+1}(-1)^n + r^n(-1)^{n+1}}{1 + 2r + r^2} = -\frac{1 + r^n}{1 + r}. \tag{3.5}$$

Combining (3.4) and (3.5), we find the following expression for the derivative $F_1(r, \theta) = \frac{\partial}{\partial r} \Re C_n(z) - \frac{\partial}{\partial r} \Re C_n(-r)$ for any odd $n \geq 1$:

$$F_1(r, \theta) = \frac{S(r, \theta)}{(1 + r)(1 - 2r\cos\theta + r^2)}, \tag{3.6}$$

where

$$S(r,\theta) = 1 - r + r^n + r^{n+2} + (1 - r - 2r^{n+1})\cos\theta + r^{n+1}(1+r)\cos n\theta - r^n(1+r)\cos(n+1)\theta. \tag{3.7}$$

Suppose that $r = \rho$, where $0 < \rho < 1$, is a critical point of $F(r,\theta)$ considered as a function of r. Then we must have $F_1(\rho,\theta) = 0$ or, equivalently, $S(\rho,\theta) = 0$. Next, we differentiate Equation (3.6) with respect to r and then evaluate the second partial derivative $F_{11}(r,\theta)$ at the critical point $r = \rho$. Then we obtain:

$$F_{11}(\rho,\theta) = \frac{S_1(\rho,\theta)}{(1+\rho)(1 - 2\rho\cos\theta + \rho^2)}. \tag{3.8}$$

Here, as above, notation $S_1(r,\theta)$ stands for the partial derivative of the function $S(r,\theta)$ with respect to its first variable. Using (3.7), we find the following explicit expression for $S_1(r,\theta)$:

$$S_1(r,\theta) = -1 + nr^{n-1} + (n+2)r^{n+1} - (1 + 2(n+1)r^n)\cos\theta + r^n(n+1+(n+2)r)\cos n\theta - r^{n-1}(n+(n+1)r)\cos(n+1)\theta. \tag{3.9}$$

Returning to a critical point $r = \rho$, we use the explicit expression for $S(r,\theta)$ given by Equation (3.7) to rewrite equation $S(\rho,\theta) = 0$ in the following equivalent form:

$$\cos n\theta = -\frac{1 - \rho + \rho^n + \rho^{n+2} + (1 - \rho - 2\rho^{n+1})\cos\theta - \rho^n(1+\rho)\cos(n+1)\theta}{\rho^{n+1}(1+\rho)}.$$

Thus, this equation expresses $\cos n\theta$ in terms of some other functions.

Substituting this expression for $\cos n\theta$ into (3.9) and replacing r with its critical value ρ, after some algebra we find that

$$\begin{aligned}\rho(1+\rho)S'(\rho) =& -1 - n - 2\rho + \rho^2 + n\rho^2 - \rho^n - 2\rho^{n+1} + \rho^{n+2} \\ & - \left(1 + n + 2\rho - \rho^2 - n\rho^2 - 2\rho^{n+2}\right)\cos\theta \\ & + \left(\rho^n + 2\rho^{n+1} + \rho^{n+2}\right)\cos(n+1)\theta \\ =& -\rho^n(1+\rho)^2\left(1 - \cos(n+1)\theta\right) \\ & - \left((1+n)(1-\rho^2) + 2\rho(1-\rho^{n+1})\right)(1+\cos\theta).\end{aligned}$$

Since the terms on the right side of the latter equation are obviously negative we conclude that $S_1(\rho,\theta) < 0$ at every possible critical point $r = \rho$ of the function $F(r,\theta)$. Now, Equation (3.8) implies that $F_{11}(\rho,\theta) < 0$ at every such critical point $r = \rho$. Therefore every critical point $r = \rho$ of $F(r,\theta)$, if it exists, must be a point of local maximum.

Next, we examine $F(r,\theta)$ at the end points of the interval $0 \le r \le 1$. We obviously have $\lim_{r\to 0^+} F(r,\theta) = 0$ for all $0 \le \theta \le \pi$. For $r = 1$ and odd $n \ge 1$, from (3.3) and (2.4) we observe that the inequality $F(1,\theta) > 0$ for $0 \le \theta < \pi$ is equivalent to the following century-old inequality:

$$\sum_{k=1}^{n} \frac{1}{k}\cos k\theta > \sum_{k=1}^{n} \frac{(-1)^k}{k}. \tag{3.10}$$

The latter inequality is a classical trigonometric inequality of W. H. Young [10] known since 1913.

Finally, since $\lim_{r\to 0^+} F(r,\theta) = 0$ and $F(1,\theta) > 0$ for all $0 \le \theta < \pi$ and since for every fixed θ, $0 \le \theta < \pi$, the function $F(r,\theta)$ considered as a function of r cannot have points of local minima, we conclude that $F(r,\theta) > 0$ for all $0 < r \le 1$ and all $0 \le \theta < \pi$. The lemma is proved. □

Summarizing our observations, which include representation (2.3), inequality (3.1), and inequality (3.2) of Lemma 2, we conclude that Theorem 2 holds true.

4. Inequalities for the modulus of Brannan's coefficients

Working with inequality (1.4) for the modulus of Brannan's coefficients, it is tempting to try the same approach based on representation (2.3), which we had used in our proof of the inequality for the real parts. Unfortunately, this approach fails because the modulus of the integrand in (2.3) does not possess the necessary maximality property. In particular, our computer experiments show that for some odd $n \ge 1$ and some values of r such that $0 < r < 1$ and r close enough to 1 the following holds:

$$\max_{0\le\theta\le\pi} |C_n(re^{i\theta})| > |C_n(-r)|.$$

This is why, while working with inequality (1.4), we have chosen to use a representation of Brannan's coefficients given by (2.5). The integrand of this representation contains the function $D_n(z)$, which is a "shift" of the coefficient $C_n(z)$; see (2.6). Our first goal is to show that an appropriate estimate of $|D_n(z)|$ would be sufficient to prove inequality (1.4).

Lemma 3. *Let n be a positive integer $0 < \alpha < 1$, and $0 < \rho \le 1$. If, for all $z = re^{i\theta}$ such that $0 < r < \rho$ and $0 \le \theta < \pi$, the shifted coefficient $D_n(z)$ satisfies the inequality $|D_n(z)| < |D_n(-|z|)|$, then*

$$|A_n(z,\alpha)| < A_n(|z|,\alpha) \tag{4.1}$$

for all $z = re^{i\theta}$ such that $0 < r \le \rho$ and $0 \le \theta < \pi$.

Proof. To prove this lemma, we first apply the triangle inequality to (2.5) and then we use our assumption that $|D_n(z)| < |D_n(-|z|)|$ for $z = re^{i\theta}$ with $0 < r < \rho$ and $0 \le \theta < \pi$. We note here that the latter inequality implies also that $|D_n(-tz)| < |D_n(-t|z|)|$ for the same z and all t, $0 < t < 1$. Then, taking into account that $\Gamma(\alpha)\Gamma(-\alpha) < 0$ and $D_n(-|z|) < 0$, we obtain the following chain of relations:

$$\begin{aligned} |A_n(z,\alpha)| \le\ & |A_n(-|z|,\alpha)| + \frac{1}{\Gamma(\alpha)|\Gamma(-\alpha)|}\int_0^1 |D_n(-tz)| t^{-\alpha-1}(1-t)^{\alpha-1}\,dt \\ <\ & A_n(-|z|,\alpha) + \frac{1}{\Gamma(\alpha)|\Gamma(-\alpha)|}\int_0^1 |D_n(-t|z|)| t^{-\alpha-1}(1-t)^{\alpha-1}\,dt \\ =\ & A_n(-|z|,\alpha) + \frac{1}{\Gamma(\alpha)\Gamma(-\alpha)}\int_0^1 D_n(-t|z|) t^{-\alpha-1}(1-t)^{\alpha-1}\,dt \\ =\ & A_n(|z|,\alpha). \end{aligned}$$

The lemma is proved. □

We have performed numerous computer experiments which lead us to the following conjecture.

Conjecture 1. *For any odd integer $n \geq 1$ and all $z = re^{i\theta}$ such that $0 < r \leq 1$ and $0 \leq \theta < \pi$, the following inequality holds:*

$$|D_n(z)| < |D_n(-|z|)|. \tag{4.2}$$

Figure 1 shows the images $l_k = D_9(L_k)$ of the circles $L_k = \{z : |z| = (11-k)/10\}$, $k = 1, \dots, 10$, under the mapping $D_9(z)$. Black dots on these images represent points which are the furthest from the origin. They all lie on the negative real axis.

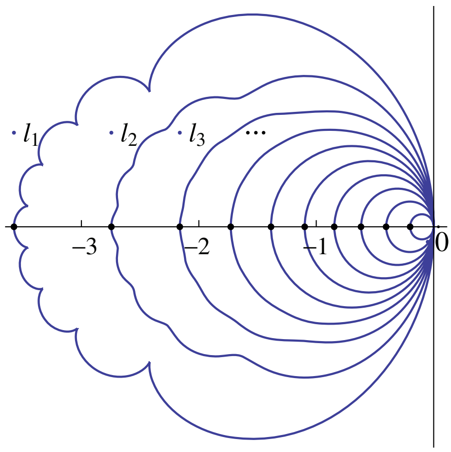

FIGURE 1. Images of the circles L_k under the mapping $D_9(z)$.

We succeeded in proving Conjecture 1 for the range $|z| \leq 1/2$.

Lemma 4. *The inequality (4.2) holds true for all odd integers $n \geq 1$ and all $z = re^{i\theta}$ such that $0 < r \leq 1/2$, $0 \leq \theta < \pi$.*

Proof. Differentiating $|D_n(re^{i\theta})|$ with respect to r and then using the inequality between the modulus and real part of a complex number, we obtain

$$\begin{aligned} \frac{\partial}{\partial r}|D_n(re^{i\theta})| = \frac{\partial}{\partial r}\, e^{\log|D_n(re^{i\theta})|} &= \Re\left(\frac{\partial}{\partial r}D_n(re^{i\theta})\frac{|D_n(re^{i\theta})|}{D_n(re^{i\theta})}\right) \\ &\leq \left|\frac{\partial}{\partial r}D_n(re^{i\theta})\right|. \end{aligned} \tag{4.3}$$

To find $\left|\frac{\partial}{\partial r}D_n(re^{i\theta})\right|$, we differentiate (2.5); then after simplification we obtain

$$\begin{aligned} \left|\frac{\partial}{\partial r}D_n(re^{i\theta})\right| &= \frac{1}{r}\left(\left(\sum_{k=1}^{n} r^k(1-\cos k\theta)\right)^2 + \left(\sum_{k=1}^{n} r^k \sin k\theta\right)^2\right)^{1/2} \\ &= \left(\frac{1}{1-r}F(r,\theta,n) + \left(\frac{1-r^n}{1-r}\right)^2\right)^{1/2}, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} F(r,\theta,n) = {}& \frac{(1-r)(1-2r^n\cos n\theta + r^{2n})}{1-2r\cos\theta + r^2} \\ &+ \frac{2(1-r^n)(r-\cos\theta + r^n\cos(n+1)\theta - r^{n+1}\cos n\theta)}{1-2r\cos\theta + r^2}. \end{aligned} \tag{4.5}$$

For $\theta = \pi$, we have $D_n(-r) = -\sum_{k=1}^{n}\left(1-(-1)^k\right)k^{-1}r^k$, which shows that $D_n(-r) < 0$ and $\frac{\partial}{\partial r}D_n(-r) < 0$ for all $r > 0$. This implies that the last relation in (4.3) holds with the sign of equality if $\theta = \pi$. Combining this observation with (4.4), we obtain

$$\frac{\partial}{\partial r}|D_n(-r)| = \left|\frac{\partial}{\partial r}D_n(-r)\right| = \left(\frac{1}{1-r}F(r,\pi,n) + \left(\frac{1-r^n}{1-r}\right)^2\right)^{1/2}. \tag{4.6}$$

We claim that for any odd integer $n \geq 3$ and real r, $0 < r \leq 1/2$, the function $F(r,\theta,n)$ considered as a function of θ achieves its maximum at $\theta = \pi$. Indeed, using (4.5), we represent the difference $F(r,\pi,n) - F(r,\theta,n)$ in the following form:

$$F(r,\pi,n) - F(r,\theta,n) = \frac{2H(r,\theta,n)}{(1+r)^2(1-2r\cos\theta+r^2)},$$

where

$$H(r,\theta,n) = c_0(1+\cos\theta) + c_1(1+\cos n\theta) + c_2(1-\cos(n+1)\theta)$$

and

$$\begin{aligned}
c_0 &= 1 - r - r^n - 4r^{n+1} + r^{n+2} + r^{2n+1} + 3r^{2n+2}\\
&= (1-r^{n+1})(1-r-r^n-3r^{n+1}),\\
c_1 &= r^n + 2r^{n+1} + r^{n+2} - r^{2n+1} - 2r^{2n+2} - r^{2n+3} = r^n(1+r)^2(1-r^{n+1}),\\
c_2 &= r^n + 2r^{n+1} + r^{n+2} - r^{2n} - 2r^{2n+1} - r^{2n+2} = r^n(1+r)^2(1-r^n).
\end{aligned}$$

Thus, $H(r,\theta,n)$ is a linear combination of positive functions $1+\cos\theta$, $1+\cos n\theta$, and $1-\cos(n+1)\theta$ depending only on n and θ with coefficients c_0, c_1, and c_2 depending only on n and r. Obviously, $c_1 > 0$ and $c_2 > 0$ for $0 < r < 1$. Furthermore, for $0 < r < 1$, the coefficient c_0 is positive if and only if $1 - r - r^n - 3r^{n+1} > 0$. The latter function is decreasing in the variable r and increasing in the variable n for $0 < r < 1$ and odd $n \geq 1$. In addition, this function is positive for $r = 1/2$ and $n = 3$. This implies that $H(r,\theta,n) > 0$ for the considered values of the parameters and therefore $F(r,\pi,n) > F(r,\theta,n)$ for these values as was claimed.

For $\theta = \pi$ and odd $n \geq 1$, Equation (4.5) gives:

$$F(r,\pi,n) = \frac{(1+r^n)(3+r-r^n-3r^{n+1})}{(1+r)^2},$$

which is obviously positive for $0 < r < 1$ and all positive integers n.

Since $F(r,\pi,n) > 0$ and $F(r,\pi,n) > F(r,\theta,n)$ for $0 < r \leq 1/2$, $0 \leq \theta < \pi$, and odd integers $n \geq 3$, it follows from Equations (4.4) and (4.6) that

$$\frac{\partial}{\partial r}|D_n(re^{i\theta})| < \frac{\partial}{\partial r}|D_n(-r)| \tag{4.7}$$

for $0 < r \leq 1/2$, $0 \leq \theta < \pi$ and all odd integers $n \geq 3$. We note here that $\lim_{r\to 0^+} D_n(re^{i\theta}) = 0$ for all $0 \leq \theta \leq \pi$. Now, integrating inequality (4.7), we conclude that for the considered values of r, θ, and n,

$$|D_n(re^{i\theta})| < |D_n(-r)|,$$

which proves the lemma. ☐

Remark. Our proof of Lemma 4 works for odd $n \geq 3$. For $n = 1$, the function $|D_n(re^{i\theta})|$ has the form $|D_n(re^{i\theta})| = |re^{i\theta} - r|$ and is obviously increasing in $0 \leq \theta \leq \pi$ for all $r > 0$.

Summarizing our results proved in Lemmas 3 and 4, we conclude that $|A_n(z, \alpha)| < A_n(|z|, \alpha)$ for odd integers $n \geq 3$ and all α and $z = re^{i\theta}$ such that $0 < \alpha < 1$, $0 < r \leq 1/2$ and $0 \leq \theta < \pi$. In the case $n = 1$, $|A_1(re^{i\theta}, \alpha)| < A_1(r, \alpha)$ holds for $0 < \alpha < 1$, $r > 0$, and $0 \leq \theta < \pi$. In particular, these results show that Theorem 1 holds true.

5. Integral representations for $\beta \neq 1$

In this section, we give two integral representations for Brannan's coefficients $A_n(\alpha, \beta, z)$ in case $0 < \beta < 1$. To derive these representations, we employ the same approach that we used to justify representation (2.3) in Section 2. Thus, we omit details and mention only that on some stages of this derivation we used the following classical identity for the Euler beta-function:

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 \tau^{a-1}(1-\tau)^{b-1}\, dt.$$

First, we represent $A_n(\alpha, \beta, z)$ as a single integral as follows:

$$\begin{aligned}
A_n(z, \alpha, \beta) &= \frac{(\beta)_n}{n!} + \sum_{k=1}^{n} \frac{(-\alpha)_k(\beta)_{n-k}}{k!(n-k)!}(-z)^k \\
&= \frac{(\beta)_n}{n!} + \sum_{k=1}^{n} \frac{(-z)^k}{k!(n-k)!} \frac{\Gamma(-\alpha+k)}{\Gamma(-\alpha)} \frac{\Gamma(\beta+n-k)}{\Gamma(\beta)} \\
&= \frac{(\beta)_n}{n!} + \frac{\Gamma(-\alpha+\beta+n)}{\Gamma(-\alpha)\Gamma(\beta)} \\
&\times \sum_{k=1}^{n} \frac{(-z)^k}{k!(n-k)!} \int_0^1 t^{-\alpha+k-1}(1-t)^{\beta+n-k-1}\, dt \\
&= \frac{(\beta)_n}{n!} + \frac{\Gamma(-\alpha+\beta+n)}{\Gamma(-\alpha)\Gamma(\beta)n!} \\
&\times \int_0^1 \left(\sum_{k=1}^{n} \binom{n}{k} \left(\frac{-tz}{1-t} \right)^k \right) t^{-\alpha-1}(1-t)^{\beta+n-1}\, dt \\
&= \frac{(\beta)_n}{n!} - \frac{\Gamma(-\alpha+\beta+n)}{\Gamma(-\alpha)\Gamma(\beta)n!} \int_0^1 P_n(z, t) t^{-\alpha-1}(1-t)^{\beta+n-1}\, dt,
\end{aligned}$$

where

$$P_n(z, t) = \left(1 - \left(1 - \frac{tz}{1-t} \right)^n \right).$$

Our second representation expresses $A_n(\alpha,\beta,z)$ as a double integral:

$$\begin{aligned}
A_n(z,\alpha,\beta) &= \frac{(\beta)_n}{n!}+\sum_{k=1}^{n}\frac{(-\alpha)_k(\beta)_{n-k}}{k!(n-k)!}(-z)^k \\
&= \frac{(\beta)_n}{n!}+\sum_{k=1}^{n}\frac{(-z)^k}{k!(n-k)!}\frac{\Gamma(-\alpha+k)}{\Gamma(-\alpha)}\frac{\Gamma(\beta+n-k)}{\Gamma(\beta)} \\
&= \frac{(\beta)_n}{n!}+\sum_{k=1}^{n}\frac{(-z)^k}{k!(n-k)!}\frac{\Gamma(-\alpha+k)\Gamma(\alpha)}{\Gamma(-\alpha)\Gamma(\alpha)}\frac{\Gamma(\beta+n-k)\Gamma(1-\beta)}{\Gamma(\beta)\Gamma(1-\beta)} \\
&= \frac{(\beta)_n}{n!}+\frac{1}{\Gamma(-\alpha)\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\beta)}\sum_{k=1}^{n}\frac{(-z)^k}{k!(n-k)!} \\
&\times\ \Gamma(k)\int_0^1 t^{-\alpha+k-1}(1-t)^{\alpha-1}\,dt \\
&\times\ \Gamma(n-k+1)\int_0^1 s^{\beta+n-k-1}(1-s)^{1-\beta-1}ds \\
&= \frac{(\beta)_n}{n!}+\frac{1}{\Gamma(-\alpha)\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\beta)} \\
&\times\int_0^1\int_0^1\left(\sum_{k=1}^{n}\frac{(-z)^k}{k}t^k s^{n-k}\right)t^{-\alpha-1}(1-t)^{\alpha-1}s^{\beta-1}(1-s)^{-\beta}dt\,ds \\
&= \frac{(\beta)_n}{n!}+\frac{1}{\Gamma(-\alpha)\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\beta)} \\
&\times\ \int_0^1\int_0^1 C_n(-zt,s)t^{-\alpha-1}(1-t)^{\alpha-1}s^{\beta-1}(1-s)^{-\beta}dt\,ds,
\end{aligned}$$

where

$$C_n(z,s)=\sum_{k=1}^{n}\frac{z^k}{k}s^{n-k}=s^nC_n(z/s).$$

We want to stress here that, even with these new representations at hand, our proof of inequality (1.5) for the real part of $A_n(\alpha,z)$ given in Section 3 does not directly extend to obtain a proof for the real part of $A_n(\alpha,\beta,z)$ if $\beta\neq 1$. The reason is that the functions $P_n(z,t)$ and $C_n(z,s)$ with $z=re^{i\theta}$ considered as functions of θ for odd n and some values of β, t, r, and s do not take their maximal values at $\theta=\pi$. However, if $r<s$, then the function $C_n(z,s)=s^nC_n(z/s)$ possesses this property because the function $C_n(z)$ does.

6. Trigonometric sums

Using the explicit expression of $C_n(z)$ given by (2.4) with $n=2m-1$, inequality (3.2) can be written as follows:

$$\sum_{k=1}^{2m-1}\frac{r^k}{k}\cos k\theta>\sum_{k=1}^{2m-1}(-1)^k\frac{r^k}{k} \tag{6.1}$$

for all integers $m\geq 1$ and real r and θ such that $0<r\leq 1$, $0\leq\theta<\pi$. The latter inequality is a generalization of the classical Young's inequality mentioned in

Section 3; see (3.10). A counterpart of (6.1) for the imaginary part of $C_n(z)$ is the following inequality

$$\sum_{k=1}^{n} \frac{r^k}{k} \sin k\theta > 0 \tag{6.2}$$

for all integers n and real r and θ such that $0 < r \leq 1$, $0 < \theta < \pi$. For $r = 1$, this, of course, is the celebrated Fejér-Jackson inequality, which was conjectured by Fejér in 1910 and proved by Jackson in 1911. For $0 < r < 1$, (6.2) follows from the Fejér-Jackson inequality and the maximum principle applied to the harmonic function $\Im C_n(z)$. Also, (6.2) is a special case of the well-known Vietoris inequality [9], which itself is a generalization of the Fejér-Jackson inequality.

Finally, using Equation (2.6), we obtain the following trigonometric form of Conjecture 1.

Conjecture 2. *For any odd integer $n \geq 1$ and all $z = re^{i\theta}$ such that $0 < r \leq 1$ and $0 \leq \theta < \pi$ the following inequality holds:*

$$\left(\sum_{k=1}^{2m-1} \frac{r^k}{k}(1 - \cos k\theta)\right)^2 + \left(\sum_{k=1}^{2m-1} \frac{r^k}{k} \sin k\theta\right)^2 < 4\left(\sum_{k=1}^{m} \frac{r^{2k-1}}{2k-1}\right)^2.$$

By Lemma 4, this conjecture holds true for $0 < r \leq 1/2$. The remaining case $1/2 < r \leq 1$ appears to be more difficult.

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