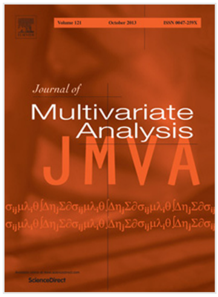

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Some New Results on the Eigenvalues of Complex Non-central Wishart Matrices with a Rank-1 Mean

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Abstract

Let $\mathbf{W}$ be an $n \times n$ complex non-central Wishart matrix with $m\ (\geq n)$ degrees of freedom and a rank-1 mean. In this paper, we consider three problems related to the eigenvalues of $\mathbf{W}$. To be specific, we derive a new expression for the cumulative distribution function (c.d.f.) of the minimum eigenvalue ($\lambda_{\min}$) of $\mathbf{W}$. The c.d.f. is expressed as the determinant of a square matrix, the size of which depends only on the difference $m-n$. This further facilitates the analysis of the microscopic limit of the minimum eigenvalue. The microscopic limit takes the form of the determinant of a square matrix with its entries expressed in terms of the modified Bessel functions of the first kind. We also develop a moment generating function based approach to derive the probability density function of the random variable $\mathrm{tr}(\mathbf{W})/\lambda_{\min}$, where $\mathrm{tr}(\cdot)$ denotes the trace of a square matrix. Moreover, we establish that, as $m, n \to \infty$ with $m-n$ fixed, $\mathrm{tr}(\mathbf{W})/\lambda_{\min}$ scales like n^3. Finally, we find the average of the reciprocal of the characteristic polynomial $\det[z\mathbf{I}_n + \mathbf{W}]$, $|\arg z| < \pi$, where $\mathbf{I}_n$ and $\det[\cdot]$ denote the identity matrix of size n and the determinant, respectively.

Keywords: Demmel condition number, Eigenvalues, Hypergeometric function of two matrix arguments, Non-central Wishart distribution, Random matrix
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1. Introduction

The eigenvalues of random matrices are known to have far-reaching implications in various scientific disciplines. Finite dimensional properties of the eigenvalues of Wishart type random matrices are of paramount importance in classical multivariate analysis (Anderson [3], Muirhead [49]), whereas recent multivariate statistical investigations have focused on establishing the asymptotic properties of the eigenvalues (Johnstone [39], Onatski et al. [52]). Various links between the eigenvalues of random matrices and statistical physics, combinatorics and integrable systems have been established over the last few decades (see e.g., Forrester [26], Mehta [47] and references therein). Apart from these areas, random matrices, especially matrices with complex Gaussian elements, have also found new applications in signal processing and wireless communications (Chiani et al. [16], Heath Jr. and Paulraj [33], Jin et al. [38], Kang and Alouini [40, 41], Maaref and Aïssa [45], Narasimhan [50], Oestges et al. [51], Ordonez et al. [53], Palomar et al. [54], Telatar [59], Tulino and Verdú [60], Zanella et al. [65]).

The majority of those studies focus on random matrix ensembles derived from zero mean Gaussian matrices. However, random matrices derived from non-zero mean Gaussian matrices have been traditionally an area of interest in multivariate analysis (Anderson [3], Constantine [17], James [37], Muirhead [49]). Moreover, mathematical objects such as zonal polynomials (Hua [36], James [37]) and hypergeometric functions of matrix arguments (Herz [34], Khatri [43]) have been introduced in multivariate analysis literature to facilitate further analysis of such non-central random matrices. Interestingly, these non-central matrices have also been referred to as random matrices with external sources in the literature of physics (Brézin and Hikami [11, 12, 13], Zinn-Justin [66]). In this respect, the classical orthogonal polynomial based characterization of the eigenvalues of random matrices (Mehta [47]) has been further extended to encompass multiple orthogonal polynomials in Bleher and Kuijlaars [8, 9]. Alternatively, capitalizing on a contour integral approach due to Kazakov (Kazakov

[42]), the authors in Ben Arous and Péché [6], Brézin and Hikami [11, 12] have introduced a double contour integral representation for the correlation kernel of the eigenvalue point process of non-central random matrices. Some recent contributions on this matter include Bassler et al. [5], Forrester [27].

One of the salient features common to those latter studies is that they exclusively focus either on spiked correlation or mean model. It is noteworthy that these two models are mathematically related to each other (Bleher and Kuijlaars [7]). As we are well aware of, the characterization of the joint eigenvalue distribution of non-central random matrices[2] involves the hypergeometric function of two matrix arguments (James [37]). It turns out that one of the argument matrices becomes reduced-rank in the presence of a spiked mean/correlation model. Specifically, when the spike is of rank one, an alternative representation of the hypergeometric function of two matrix arguments has recently been discovered independently by Mo (Mo [48]), Wang (Wang [62]) and Onatski (Onatski et al. [52]). The key contribution there amounts to the representation of the hypergeometric function of two matrix arguments with a rank-1 argument matrix in terms of an infinite series involving a single contour integral. This representation has been subsequently used to further characterize the asymptotic behaviors of the eigenvalues of non-central random matrices (Mo [48], Wang [62]). Further generalization of the contour integral representation given in Mo [48], Wang [62], Onatski et al. [52] to an arbitrary hypergeometric function with two matrix arguments having a rank-1 argument matrix has been reported in Dharmawansa and Johnstone [22].

In this paper, we analyze three problems pertaining to the eigenvalues of a finite dimensional complex non-central Wishart matrix with a rank-1 mean matrix[3]. Let $0 < \lambda_1 \leq \cdots \leq \lambda_n$ be the ordered eigenvalues of an $n \times n$ complex

[2]Here the term "non-central random matrices" refers to non-central Gaussian and Wishart matrices.

[3]This is also known as the shifted mean chiral Gaussian ensemble with $\beta = 2$ (i.e., the complex case) (Forrester [27]).

non-central Wishart matrix $\mathbf{W}$ with m degrees of freedom and a rank-1 mean. We are interested in the following three problems.

1. The characterization of the cumulative distribution function (c.d.f.) of the minimum eigenvalue of $\mathbf{W}$ as the determinant of a square matrix, the size of which depends on the difference of the number of degrees of freedom and n (i.e., $m-n$).
2. The statistical characterization of the random quantity $\text{tr}(\mathbf{W})/\lambda_1$ with $\text{tr}(\cdot)$ denoting the trace of a square matrix.
3. The statistical average of the reciprocal of the characteristic polynomial $\det[z\mathbf{I}_n + \mathbf{W}]$, $|\arg z| < \pi$, with $\det[\cdot]$ and $\mathbf{I}_n$ denoting the determinant of a square matrix and the $n \times n$ identity matrix, respectively.

The above quantities have found many applications in contemporary wireless communications systems. In particular, complex non-central Wishart matrices with a rank-one mean arise in multiple-input multiple-output (MIMO) channels characterized by strong line-of-sight components (i.e., Rician fading) (Kang and Alouini [40, 41]). Therefore, the functionals of the eigenvalues of complex non-central Wishart matrices with a rank one mean have been instrumental in MIMO signal processing (Jin et al. [38], Kang and Alouini [40, 41], Maaref and Aïssa [45], Zanella et al. [65]). For example, the distribution of the minimum eigenvalue is important in characterizing the performance of MIMO multichannel beamforming (MB) systems[4] (Narasimhan [50], Ordonez et al. [53], Palomar et al. [54], Jin et al. [38]). Specifically, the global performance of an MB system is dominated by the performance of the weakest link (i.e., the link corresponding to the minimum eigenmode transmission). The quantity $\text{tr}(\mathbf{W})/\lambda_1$ is of paramount importance in the designing of adaptive multiantenna transmission techniques (Heath Jr. and Paulraj [33]) and the modeling of physical multiantenna transmission channels (Oestges et al. [51]). Moreover, condition numbers of this form

[4] MIMO MB systems are also known as spatial multiplexing MIMO systems with CSI (channel state information) in the literature (Ordonez et al. [53]).

have been introduced to mutiple antenna spectrum sensing in cognitive radio (see e.g., Debbah and Couillet [19] and references therein). More recent applications of the eigenvalues of Wishart matrices include the design and analysis of massive MIMO systems (see e.g., Hoydis et al. [35] and references therein).

The first problem has a straightforward solution in the form of the determinant of a square matrix of size $n \times n$ (Jin et al. [38], Maaref and Aïssa [45], McKay [46], Zanella et al. [65]). This stems from the determinant representation of the hypergeometric function of two matrix arguments due to Khatri (Khatri [43]) (see also Gross and Richards [32]). However, in certain cases, it is convenient to have an expression with the determinant of a square matrix of size $m - n$. Therefore, in this work, by leveraging the knowledge of classical orthogonal polynomials, we derive an alternative expression for the c.d.f. of the minimum eigenvalue which involves the determinant of a square matrix of size $m - n + 1$. This new form is highly desirable when the difference between m and n is small irrespective of their individual magnitudes. In such a situation, this new expression circumvents the analytical complexities associated with the above straightforward solution which requires the evaluation of the determinant of an $n \times n$ square matrix. This key representation, in turn, facilitates the further analysis of the so-called microscopic limit of the minimum eigenvalue (i.e., the limit when $m, n \to \infty$ such that $m - n$ is fixed) which is known to have a Fredholm determinant representation (Ben Arous and Péché [6]). Our results reveal that this microscopic limit coincides with the corresponding limit in the central Wishart case.

The random quantity of our interest in the second problem is commonly known as the Demmel condition number in the literature of numerical analysis (Demmel [20]). As opposed to the case corresponding to the central Wishart matrices (Muirhead [49]), $\mathrm{tr}(\mathbf{W})$ and $\lambda_1/\mathrm{tr}(\mathbf{W})$ are no longer statistically independent. Furthermore, a direct Laplace transform relationship between $\lambda_1/\mathrm{tr}(\mathbf{W})$ and the probability density of the minimum eigenvalue of $\mathbf{W}$ has not been reported in the literature. However, such a relationship among these random quantities in the case of central Wishart matrices has been reported in Dharmawansa et al. [24], Krishnaiah and Schuurmann [44]. Therefore, we introduce a moment generating function (m.g.f.) based framework to solve the second problem. In particular, using a classical orthogonal polynomial approach, we derive the m.g.f. of the random variable of our interest in terms of a single integral involving the determinant of a square matrix of size $m-n+1$. Upon taking the direct Laplace inversion of the m.g.f. we then obtain an exact expression for the probability density function (p.d.f.). The remarkable fact of having the determinant of a square matrix of size $m-n+1$ makes it suitable to be used when the relative difference between m and n is small. For instance, in the special case of $m=n$, the p.d.f. simplifies to an expression involving a single infinite summation. Moreover, we have determined the asymptotic scaled limit of $\text{tr}(\mathbf{W})/\lambda_1$ as $m,n \to \infty$ with $m-n$ fixed. It turns out that, under the above asymptotic setting, the random quantity $\text{tr}(\mathbf{W})/\lambda_1$ scales like n^3.

Statistical averages akin to the third problem are closely related to some problems of the classical number theory (see e.g., Borodin and Strahov [10] for a comprehensive list of references in this respect). A generalized framework based on the duality between certain matrix ensembles has been proposed in Desrosiers [21] to solve certain problems involving the averages of the reciprocals of characteristic polynomials pertaining to non-central Wishart matrices. However, the third problem of our interest does not seem to be consistent with that framework, since the specific parameters associated with our problem do not satisfy the requirements in [21]. This particular problem has not been addressed in a more recent work by Forrester (Bassler et al. [5]) on the averages of characteristic polynomials for shifted mean chiral Gaussian ensembles. Therefore, again following the classical orthogonal polynomial approach, here we derive a new expression for this particular average. The resultant expression turns out to have a single infinite series. This is not surprising, since in the case of a central Wishart matrix the corresponding answer depends only on the number of characteristic polynomials rather than the size of the random matrix (Borodin and Strahov [10], Desrosiers [21], Fyodorov [29], Fyodorov and Strahov [30]).

The rest of this paper is organized as follows. We begin Section 2 by deriving a new p.d.f. for the eigenvalues of a complex non-central Wishart matrix with a rank-1 mean. In Section 3, we use the new joint eigenvalue p.d.f. to derive the c.d.f. of the minimum eigenvalue in terms of the determinant of a square matrix of size $m-n+1$. We also establish the microscopic limit of the minimum eigenvalue. Section 4 addresses the problem of statistical characterization of the random quantity $\mathrm{tr}(\mathbf{W})/\lambda_1$ by deriving the corresponding m.g.f. and p.d.f. expressions. Moreover, we show that, as $m, n \to \infty$ with $m-n$ fixed, the random variable $\mathrm{tr}(\mathbf{W})/\lambda_1$ scales like n^3. Section 5 derives the average of the reciprocal of the characteristic polynomial $\det[z\mathbf{I}_n + \mathbf{W}]$, $|\arg z| < \pi$.

2. Preliminaries

Let us first present some results related to the p.d.f. of a complex non-central Wishart matrix. In what follows, we use $(\cdot)^*$ to denote the conjugate transpose of a matrix and $||\mathbf{A}||_F^2$ to represent $\mathrm{tr}(\mathbf{A}^*\mathbf{A})$ where $\mathbf{A} \in \mathbb{C}^{m\times n}$. Moreover, we use $\mathrm{E}_{\mathbf{A}}(\cdot)$ to denote the mathematical expectation with respect to $\mathbf{A}$.

Theorem 1. *Let $\mathbf{X} \in \mathbb{C}^{m\times n}$ be distributed as $\mathcal{CN}_{n,m}(\mathbf{M}, \mathbf{I}_m \otimes \mathbf{I}_n)$ where $\mathbf{M} \in \mathbb{C}^{n\times n}$ with $m \geq n$. Then $\mathbf{W} = \mathbf{X}^*\mathbf{X}$ has a complex non-central Wishart distribution $\mathcal{W}_n(m, \mathbf{I}_n, \mathbf{M}^*\mathbf{M})$ with p.d.f. (James [37])*

$$f_{\mathbf{W}}(\mathbf{W}) = \frac{e^{-\mathrm{tr}(\mathbf{M}^*\mathbf{M})}}{\widetilde{\Gamma}_n(m)} (\det[\mathbf{W}])^{m-n} e^{-\mathrm{tr}(\mathbf{W})} {}_0\widetilde{F}_1(m; \mathbf{M}^*\mathbf{M}\mathbf{W}) \tag{1}$$

where $\widetilde{\Gamma}_n(m) = \pi^{\frac{m(m-1)}{2}} \prod_{i=1}^{n} \Gamma(m-i+1)$ and ${}_0\widetilde{F}_1(\cdot;\cdot)$ denotes the complex hypergeometric function of one matrix argument. In particular, for a Hermitian positive definite $n \times n$ matrix $\mathbf{A}$, we have (James [37])

$${}_0\widetilde{F}_1(p; \mathbf{A}) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \frac{C_\kappa(\mathbf{A})}{[p]_\kappa}$$

where $C_\kappa(\cdot)$ is the complex zonal polynomial[5]*, $\kappa = (k_1, \ldots, k_n)$, with k_i's being non-negative integers, is a partition of k such that $k_1 \geq \cdots \geq k_n \geq 0$ and*

[5]The zonal polynomial $C_\kappa(\mathbf{A})$ is a symmetric, homogeneous polynomial of degree k in the eigenvalues of $\mathbf{A}$. However, the specific definition of the zonal polynomial is not given here

$\sum_{i=1}^{n} k_i = k$. *Also* $[n]_\kappa = \prod_{i=1}^{n}(n-i+1)_{k_i}$ *with* $(a)_n = a(a+1)\cdots(a+n-1)$ *denoting the Pochhammer symbol.*

The following theorem is due to James (James [37]).

Theorem 2. *The joint density of the ordered eigenvalues* $0 < \lambda_1 \leq \cdots \leq \lambda_n$ *of* $\mathbf{W}$ *is given by James [37]*

$$f_{\mathbf{\Lambda}}(\lambda_1, \ldots, \lambda_n) = K_{m,n} e^{-\mathrm{tr}(\mathbf{M}^*\mathbf{M})} \Delta_n^2(\boldsymbol{\lambda}) \prod_{i=1}^{n} \lambda_i^{m-n} e^{-\lambda_i} {}_0\widetilde{F}_1(m; \mathbf{\Lambda}, \mathbf{M}^*\mathbf{M}) \quad (2)$$

where

$$K_{m,n} = \frac{1}{\prod_{i=1}^{n} \Gamma(m-i+1)\Gamma(n-i+1)},$$

$\mathbf{\Lambda} = \mathrm{diag}(\boldsymbol{\lambda})$ *with* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ *and* $\Delta_n(\boldsymbol{\lambda}) = \prod_{1 \leq i < k \leq n} (\lambda_k - \lambda_i)$. *Moreover,* ${}_0\widetilde{F}_1(\cdot; \cdot, \cdot)$ *denotes the complex hypergeometric function of two matrix arguments. For Hermitian positive definite* $n \times n$ *matrices* $\mathbf{A}$ *and* $\mathbf{B}$, *we have*

$$ {}_0\widetilde{F}_1(m; \mathbf{A}, \mathbf{B}) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \frac{C_\kappa(\mathbf{A}) C_\kappa(\mathbf{B})}{[m]_\kappa C_\kappa(\mathbf{I}_n)}.$$

Since we are interested in a rank-1 mean matrix $\mathbf{M}$, we can further simplify the above expression for the joint density of the eigenvalues to obtain

$$f_{\mathbf{\Lambda}}(\lambda_1, \ldots, \lambda_n) = \mathcal{K}_{n,\alpha} \frac{e^{-\mu}}{\mu^{n-1}} \prod_{i=1}^{n} \lambda_i^{\alpha} e^{-\lambda_i} \Delta_n^2(\boldsymbol{\lambda}) \sum_{k=1}^{n} \frac{{}_0F_1(\alpha+1; \mu\lambda_k)}{\prod_{\substack{i=1 \\ i \neq k}}^{n} (\lambda_k - \lambda_i)} \quad (3)$$

where $\mu = \mathrm{tr}(\mathbf{M}^*\mathbf{M})$ and

$$\mathcal{K}_{n,\alpha} = K_{n+\alpha,n} \frac{(n-1)!(n+\alpha-1)!}{\alpha!}$$

165 with $\alpha = m - n$.

Remark 1. One can use either the contour integral approach given in Wang [62] or the repeated application of the l'Hospital's rule due to Khatri (Khatri

as it is not required in the subsequent analysis. More details of the zonal polynomials can be found in James [37], Takemura [58].

[43])[6] to obtain the above form. Since the algebra involved in both approaches are fairly standard and straightforward, we do not show the specific steps of the derivation of (3). However, we feel that the contour integral approach is the most transparent way of obtaining the above expression.

It is important to note that the functional form given in (3) facilitates the use of classical orthogonal polynomial approach in solving the three problems of our interest.

Remark 2. A different normalization scheme has been employed in contemporary signal processing literature to characterize the MIMO Rician channel (e.g., see Jin et al. [38], Kang and Alouini [40, 41], Maaref and Aïssa [45], Zanella et al. [65] and references therein). In particular, the matrix $\mathbf{X}$ is normalized as

$$\mathbf{X} = \sqrt{\frac{K}{K+1}}\mathbf{X}_d + \sqrt{\frac{1}{K+1}}\mathbf{X}_r \tag{4}$$

where $\mathbf{X}_r \sim \mathcal{CN}_{m,n}(\mathbf{0}, \mathbf{I}_m \otimes \mathbf{I}_n)$, $\mathbf{X}_d \in \mathbb{C}^{m\times n}$ is the deterministic component normalized such that $||\mathbf{X}_d||_F^2 = mn$ and K is the Rician factor. Clearly, the above normalization gives

$$\mathrm{E}\left(||\mathbf{X}||_F^2\right) = mn.$$

Since we can rewrite the above model as

$$\mathbf{X} = \sqrt{\frac{1}{K+1}}\left[\sqrt{K}\mathbf{X}_d + \mathbf{X}_r\right],$$

the eigenvalue spectrum of $\mathbf{X}^*\mathbf{X}$ takes the form

$$\lambda_i\left(\mathbf{X}^*\mathbf{X}\right) = \frac{1}{K+1}\lambda_i\left(\mathbf{S}\right), \quad i = 1, \dots, n \tag{5}$$

where

$$\mathbf{S} \sim \mathcal{W}_n(m, \mathbf{I}_n, K\mathbf{X}_d^*\mathbf{X}_d).$$

[6]Repeated application of the l'Hospital's rule in the context of simplifying indeterminate forms involving determinants is given in Khatri [43].

Moreover, for rank-one $\mathbf{X}_d$, we find

$$\mu = K\mathrm{tr}(\mathbf{X}_d^*\mathbf{X}_d) = Kmn. \tag{6}$$

Therefore, in view of (5) and (6), the subsequent results developed in this paper can readily be applied to the rank-one MIMO Rician channel.

Let us now see how to derive a new expression for the c.d.f. of the minimum eigenvalue of a complex non-central Wishart matrix with a rank-1 mean starting from the joint p.d.f. given above.

Before proceeding, it is worth mentioning the following useful results.

Definition 3. For $\rho > -1$, the generalized Laguerre polynomial of degree M, $L_M^{(\rho)}(z)$, is given by Szegö [57]

$$L_M^{(\rho)}(z) = \frac{(\rho+1)_M}{M!}\sum_{j=0}^{M}\frac{(-M)_j}{(\rho+1)_j}\frac{z^j}{j!}, \tag{7}$$

with the kth derivative satisfying

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}L_M^{(\rho)}(z) = (-1)^k L_{M-k}^{(\rho+k)}(z). \tag{8}$$

Also $L_M^{(\rho)}(z)$ satisfies the following contiguous relationship

$$L_M^{(\rho-1)}(z) = L_M^{(\rho)}(z) - L_{M-1}^{(\rho)}(z). \tag{9}$$

Definition 4. For a negative integer $-M$, we have the following relation

$$(-M)_j = \begin{cases} (-1)^j\frac{M!}{(M-j)!} & \text{for } j \leq M \\ 0 & \text{for } j > M. \end{cases} \tag{10}$$

Lemma 3. *Following Gradshteyn and Ryzhik [31, Eq. 7.414.7] and Andrews et al. [4, Corollary 2.2.3], for* $j, k \in \{0, 1, 2, \ldots\}$, *we can establish*

$$\int_0^\infty x^j e^{-x} L_M^{(k)}(x)\mathrm{d}x = \frac{j!}{M!}(k-j)_M.$$

The following compact notation has been used to represent the determinant of an $M \times M$ block matrix:

$$\det \left[a_{i,1} \quad b_{i,j-1}\right]_{\substack{i=1,\ldots,M \\ j=2,\ldots,M}} = \begin{vmatrix} a_{1,1} & b_{1,1} & b_{1,2} & \cdots & b_{1,M-1} \\ a_{2,1} & b_{2,1} & b_{2,2} & \cdots & b_{2,M-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{M,1} & b_{M,1} & b_{M,2} & \cdots & b_{M,M-1} \end{vmatrix}.$$

3. Cumulative Distribution of the Minimum Eigenvalue

Here we derive a new expression for the c.d.f. of the minimum eigenvalue $\lambda_{\min}$ of $\mathbf{W}$ with a rank-1 mean.

By definition, the c.d.f. of $\lambda_{\min}$ is given by

$$F_{\lambda_{\min}}(x) = \Pr\left(\lambda_1 < x\right) = 1 - \Pr\left(\lambda_1 \geq x\right) \tag{11}$$

where

$$\Pr\left(\lambda_1 \geq x\right) = \int_{x \leq \lambda_1 \leq \cdots \leq \lambda_n < \infty} f_{\mathbf{\Lambda}}\left(\lambda_1, \ldots, \lambda_n\right) \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n. \tag{12}$$

The following theorem gives the c.d.f. of $\lambda_{\min}$.

Theorem 4. *Let* $\mathbf{W} \sim \mathcal{W}_n\left(m, \mathbf{I}_n, \mathbf{M}^*\mathbf{M}\right)$*, where* $\mathbf{M}$ *is rank-*1 *and* $\mathrm{tr}(\mathbf{M}^*\mathbf{M}) = \mu$*. Then the c.d.f. of the minimum eigenvalue of* $\mathbf{W}$ *is given by*

$$F_{\lambda_{\min}}(x) = 1 - (n+\alpha-1)!\, e^{-nx} \det\left[(-\mu)^{i-1}\psi_i(\mu, x) \quad L_{n+i-j}^{(j-2)}(-x)\right]_{\substack{i=1,\ldots,\alpha+1 \\ j=2,\ldots,\alpha+1}} \tag{13}$$

where $\alpha = m - n$*,*

$$\psi_i(\mu, x) = \frac{1}{(\alpha+i+n-2)!} \sum_{k=0}^{\infty} \frac{(x\mu)^k {}_1F_1\left(\alpha+k; \alpha+n+i+k-1; -\mu\right)}{k!(\alpha+i+n-1)_k},$$

185 *and* ${}_1F_1(a; c; z)$ *is the confluent hypergeometric function of the first kind.*

Proof. See Appendix A.

Remark 5. Alternatively, we can express $\psi_i(\mu, x)$ as

$$\psi_i(\mu, x) = \frac{e^{-\mu}}{(\alpha+i+n-2)!} \Phi_3\left(n+i-1, n+\alpha+i-1; \mu, x\mu\right) \tag{14}$$

where $\Phi_3(a,c;x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(a)_i x^i y^j/(c)_{i+j} i! j!$ is the confluent hypergeometric function of two variables (Erdélyi [25, Eq. 5.7.1.23]).

In the special case of $\alpha = 0$ (i.e., $m = n$), (13) admits the following simple form

$$\begin{aligned} F_{\lambda_{\min}}(x) &= 1 - e^{-nx}\sum_{k=0}^{\infty}\frac{(x\mu)^k}{k!(n)_k}{}_1F_1(k;n+k;-\mu) \\ &= 1 - e^{-\mu-nx}\Phi_3(n,n;\mu,x\mu) \end{aligned} \tag{15}$$

which coincides with what we have derived in Dharmawansa and McKay [23, Eq. 32/39] purely based on a matrix integral approach[7].

In addition, it is not difficult to show that, for $\mu = 0$, (13) simplifies to Forrester and Hughes [28, Eq. 3.19]

$$F_{\lambda_{\min}}(x) = 1 - e^{-nx}\det\left[L_{n+i-j}^{(j-1)}(-x)\right]_{i,j=1,\ldots,\alpha}. \tag{16}$$

It is worth noting that (13) provides an efficient way of evaluating the c.d.f. of $\lambda_{\min}$ particularly for small values of α. Moreover, since the algebraic complexity depends only on n and the difference of m and n (i.e., $m-n$), this result is instrumental in evaluating the microscopic limit of $\lambda_{\min}$.

Alternative expressions[8] for the c.d.f. of $\lambda_{\min}$ have been reported in Jin et al. [38, Eq. 15], Maaref and Aïssa [45, Eq. 18]. and Zanella et al. [65, Eq. 11]. The key difference between our result and those alternative expressions is that the latter involve the determinants of $n \times n$ matrices. Therefore, they are not amenable to asymptotic analysis as m and n grow large, but their difference does not.

Figure 1 compares the analytical c.d.f. result for the minimum eigenvalue of non-central Wishart matrix with simulated data. Analytical curves are computed based on Theorem 4. As can be seen from the figure, our analytical

[7]Since the results given in Dharmawansa and McKay [23] are valid for an arbitrary covariance matrix with $\alpha = 0$, one has to assume the identity covariance to obtain the above results.

[8]These expressions are valid for an arbitrary rank mean matrix.

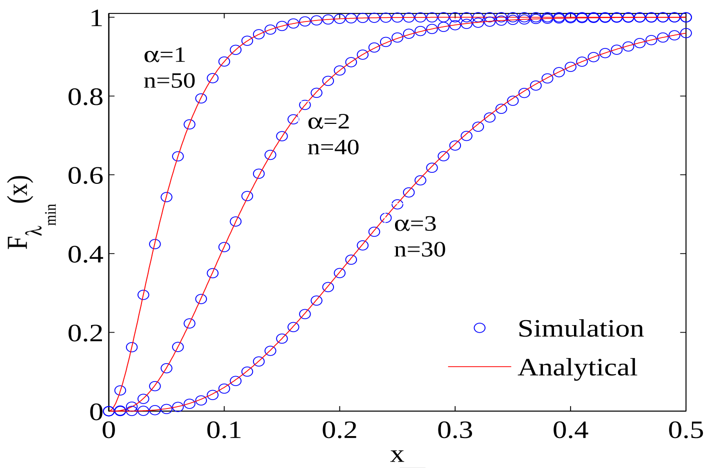

Figure 1: Comparison of simulated data points and the analytical c.d.f.s of $\lambda_{\min}$ for various values of n and α with $\mu = 10$.

results match with the simulated data thus validating our theorem. We note that the infinite summation in (13) has been truncated to 5 terms in each of the calculation. This fact in turn demonstrates the fast convergence of the given infinite series.

Figures 2, 3 and 4 further demonstrate the effects of n, α and μ, respectively, on the c.d.f. of $\lambda_{\min}$ for different parameter configurations.

For certain applications (Wang and Giannakis [63]), the behavior of the c.d.f. of $\lambda_{\min}$ at the origin is of paramount importance. Therefore, fig. 5 further demonstrates that particular behavior. Here the infinite series (13) has been truncated to 10 terms.

Remark 6. The dynamics of fig. 4 suggests that the increasing μ increases the $\lambda_{\min}$. However, it is noteworthy that for rank-one MIMO Rician channel, the $\lambda_{\min}$ decreases with the increasing K (Jin et al. [38], Kang and Alouini [40, 41]). This inconsistency between the two models can be explained using the relations (5) and (6).

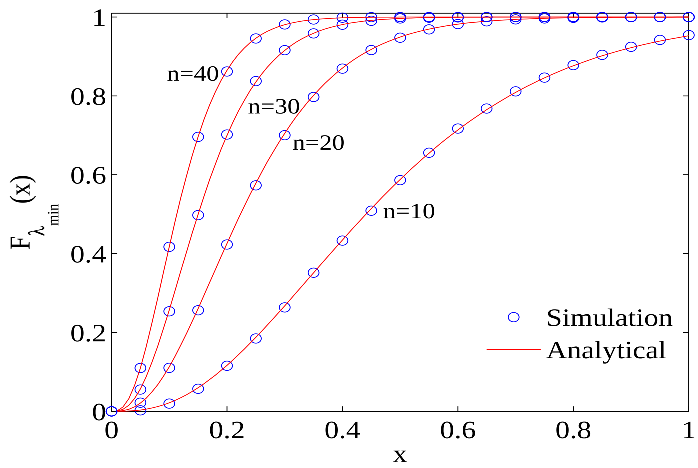

Figure 2: Illustration of the behavior of the c.d.f. of $\lambda_{\min}$ for various values of n. The results are shown for $\mu = 10$ and $\alpha = 2$.

3.1. Asymptotic Characterization (Microscopic Limit) of the Smallest Eigenvalue

Here we investigate the so called microscopic limit of the c.d.f. of $\lambda_{\min}$. To be precise, we would consider the c.d.f. of suitably scaled $\lambda_{\min}$ for fixed α when $m, n \to \infty$. The following corollary gives the microscopic limit.

Corollary 5. *As m and n tend to ∞ such that $\alpha = m - n$ is fixed, the scaled minimum eigenvalue $n\lambda_1$ converges in distribution to a random variable X with the c.d.f. $F_X(x)$. In particular, we have*

$$\lim_{n\to\infty} F_{n\lambda_1}(x) = F_X(x) = 1 - e^{-x} \det\left[I_{i-j}(2\sqrt{x})\right]_{i,j=1,\ldots,\alpha}. \tag{17}$$

Proof. See Appendix B .

Clearly, the above asymptotic expression does not depend on μ. Moreover, this limiting c.d.f. coincides with the limiting c.d.f. corresponding to central Wishart case in Forrester and Hughes [28, Eq. 3.33].

The advantage of the asymptotic formula given in corollary 5 is that it provides an easy to use expression which compares favorably with finite n results.

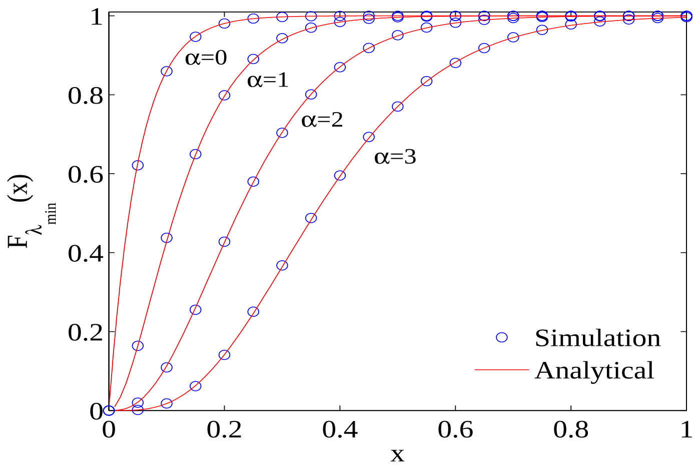

Figure 3: Illustration of the behavior of the c.d.f. of $\lambda_{\min}$ for various values of α. The results are shown for $\mu = 10$ and $n = 20$.

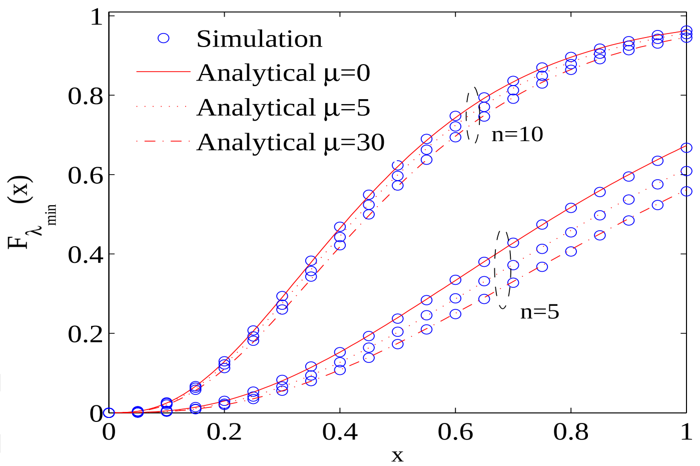

Figure 4: Illustration of the behavior of the c.d.f. of $\lambda_{\min}$ for various values of μ. The results are shown for $n = 5$ and $n = 10$ with $\alpha = 2$.

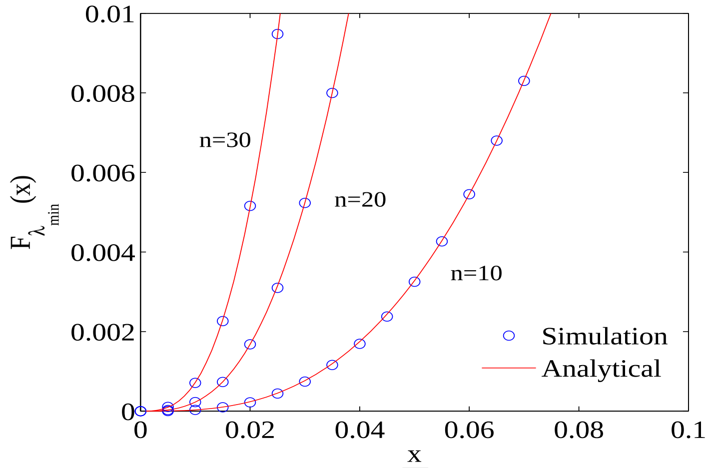

Figure 5: Illustration of the behavior of the c.d.f. of $\lambda_{\min}$ at the origin for various values of n. The results are shown for $\mu = 10$ and $\alpha = 2$.

230 To further highlight this fact, in Fig. 6, we compare the analytical asymptotic p.d.f. derived in corollary 5 with simulated data points.

Having analyzed the behavior of the minimum eigenvalue of $\mathbf{W}$, let us now move on to determine the distribution of the random variable $\mathrm{tr}(\mathbf{W})/\lambda_1$.

4. The Distribution of $\mathrm{tr}(\mathbf{W})/\lambda_1$

Here we study the distribution of the quantity

$$V = \frac{\mathrm{tr}(\mathbf{W})}{\lambda_1} = \frac{\sum_{j=1}^{n} \lambda_j}{\lambda_1}. \tag{18}$$

It turns out that this quantity is intimately related to the distribution of the minimum eigenvalue of $\mathbf{W}$ given the constraint $\mathrm{tr}(\mathbf{W}) = 1$ (i.e., fixed trace) (Chen et al. [15]). To be precise, the latter is distributed as $1/V$. Apart from that, the most notable application of the distribution of V is the so-called "smoothed analysis of condition numbers" (Spielman and Teng [56]). For a given function $g : \mathbb{C}^{m\times n} \to \mathbb{R}_+$ (e.g., the 2−norm condition number), $\mathbf{A} \sim \mathcal{CN}_{m,n}(\mathbf{M}, \sigma^2\mathbf{I}_m \otimes \mathbf{I}_n)$ with $0 < \sigma \leq 1$ and $\mathbf{M} \in \mathbb{C}^{m\times n}$ being arbitrary such that either $\mathrm{tr}(\mathbf{M}^*\mathbf{M}) = 1$ or $||\mathbf{M}||_2 \leq \sqrt{n}$ is satisfied, under the smoothed

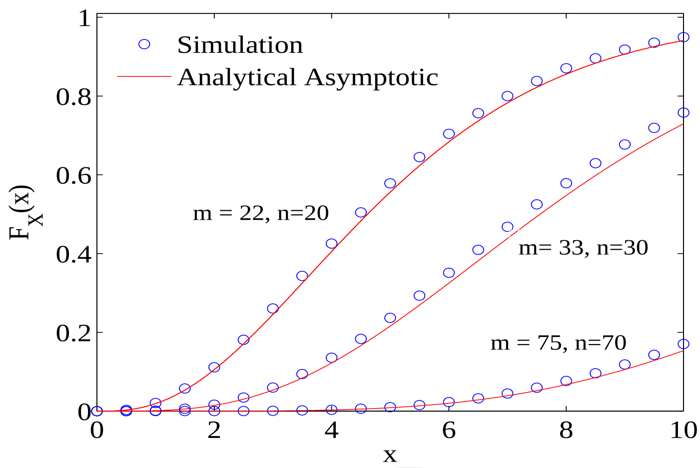

Figure 6: Comparison of the analytical asymptotic c.d.f. of $\lambda_{\min}$ and simulated data points for $\mu = 10$.

analysis framework, a typical problem is to study the behavior of (Wschebor [64], Sankar et al. [55], Cucker et al. [18], Bürgisser et al. [14])

$$\sup_{\mathbf{M}} \mathrm{E}_{\mathbf{A}} \left(g(\mathbf{A}) \right) \tag{19}$$

where $||\cdot||_2$ denote the 2-norm. For mathematical tractability, sometimes it is assumed that the matrix $\mathbf{M}$ is of rank one [64]. Bounds on the quantity (19) have been derived in the literature when $g(\mathbf{A})$ defines various condition numbers (see, e.g., Sankar et al. [55], Cucker et al. [18], Bürgisser et al. [14] and references therein). Among those condition numbers, the one introduced by James Demmel (Demmel [20]) plays an important role in understanding the behaviors of other condition numbers arising in different contexts. For a rectangular matrix $\mathbf{X} \in \mathbb{C}^{m\times n}$, the function g defined by [18]

$$g(\mathbf{X}) = ||\mathbf{X}||_F ||\mathbf{X}^{\dagger}||_2 \tag{20}$$

with $\mathbf{X}^\dagger$ denoting the Moore-Penrose inverse, gives the Demmel condition number[9]. In particular, for the matrix of our interest $\mathbf{X} \sim \mathcal{CN}_{m,n}(\mathbf{M}, \mathbf{I}_m \otimes \mathbf{I}_n)$ with $m \geq n$, (20) specializes to Dharmawansa et al. [24]

$$g(\mathbf{X}) = \sqrt{\frac{\sum_{j=1}^{n} \lambda_j}{\lambda_1}} = \sqrt{V}$$

where $\lambda_1 \leq \cdots \leq \lambda_n$ are the ordered eigenvalues of $\mathbf{W} = \mathbf{X}^*\mathbf{X}$. In light of these developments we can clearly see that the distribution of V is of great importance in performing the smoothed analysis on the Demmel condition number.

Having understood the importance of the variable V in (18), we now focus on deriving its p.d.f. when the matrix $\mathbf{W}$ has a rank-1 mean. For this purpose, here we adopt an approach based on the m.g.f. of V. We have the following key result.

Theorem 6. *Let $\mathbf{W} \sim \mathcal{W}_n\left(m, \mathbf{I}_n, \mathbf{M}^*\mathbf{M}\right)$, where $\mathbf{M}$ is rank-1 and $\mathrm{tr}(\mathbf{M}^*\mathbf{M}) = \mu$. Then the p.d.f. of V is given by*

$$f_V^{(\alpha)}(v) = (n-1)! \frac{e^{-\mu}}{v^{n(n+\alpha)}} \mathcal{L}^{-1}\left\{ \frac{e^{-ns}}{s^{(n-1)(n+\alpha+1)}} R(s,v,\mu) \right\} \tag{21}$$

where

$$R(s,v,\mu) = \det\left[\left(-\frac{\mu}{sv}\right)^{i-1} \phi_i(\mu,s,v) \quad L^{(j)}_{n+i-1-j}(-s) \right]_{\substack{i=1,\ldots,\alpha+1 \\ j=2,\ldots,\alpha+1}}$$

$$\phi_i(\mu,s,v) = \sum_{k=0}^{\infty} \frac{a_i(k)}{k!} \left(\frac{\mu}{sv}\right)^k {}_1F_1\left(n^2+n\alpha+k+i-1; n+i+k+\alpha-1; \frac{\mu}{v}\right)$$

$$a_i(k) = (n+i-1)\frac{(n^2+n\alpha+i-2)!}{(n+i+\alpha-2)!} \frac{(n+i)_k (n+i-2)_k (n^2+n\alpha+i-1)_k}{(n+i-1)_k (n+i+\alpha-1)_k}$$

and $\mathcal{L}^{-1}(\cdot)$ denotes the inverse Laplace transform.

Proof. See Appendix C.

[9] This generalizes the condition number definition given in Demmel [20] to $m \times n$ rectangular matrices.

Although further simplification of (21) seems intractable for general matrix dimensions m and n, we can obtain a relatively simple expression in the important case of square matrices (i.e., $m = n$), which is given in the following corollary.

Corollary 7. *For $\alpha = 0$, (21) becomes*

$$f_V^{(0)}(v) = n(n^2-1)e^{-\mu}(v-n)^{n^2-2}v^{-n^2} \times \sum_{k=0}^{\infty} \frac{(n^2)_k}{(n)_k k!}\left(\frac{\mu}{v}\right)^k {}_3F_3\left(n+1, n-1, n^2+k; n, n+k, n^2-1; \mu\left(1-\frac{n}{v}\right)\right) \times H(v-n) \quad (22)$$

where $H(z)$ denotes the unit step function and ${}_3F_3(a_1, a_2, a_3; c_1, c_2, c_3; z)$ is the generalized hypergeometric function (Erdélyi [25]).

We also note that, for $\mu = 0$ (i.e., when $\mathbf{W}$ is a central Wishart matrix), (21) simplifies to

$$f_V^{(\alpha)}(v) = \frac{n!(n^2+n\alpha-1)!}{(n+\alpha-1)!v^{n(n+\alpha)}} \times \mathcal{L}^{-1}\left\{\frac{e^{-ns}}{s^{(n-1)(n+\alpha+1)}} \det\left[L_{n+i-j-1}^{(j+1)}(-s)\right]_{i,j=1,\ldots,\alpha}\right\} \quad (23)$$

which coincides with the corresponding result given in Dharmawansa et al. [24, Corollary 3.2].

Figure 7 compares the analytical p.d.f. $f_V^{(0)}$ derived in Corollary 7 with simulated data points corresponding to $n = 5, 7$ and 10. The agreement between the analytical and simulation results is clearly evident from the figure. Note that we have used as few as 4 terms in the infinite summation (22) in each calculation. Moreover, figures 8 and 9 demonstrate the effects of μ on the p.d.f. of V corresponding to $\alpha = 0$ case. Although we have a wide range of μ values, we have used only 4 terms in the infinite summation in each calculation.

Remark 7. Since the random quantity V is scale invariant and for fixed m, n, $\mu \propto K$, it is expected that the behavior of V for different K with respect to rank-one MIMO Rician model is consistent with figs. 8 and 9.

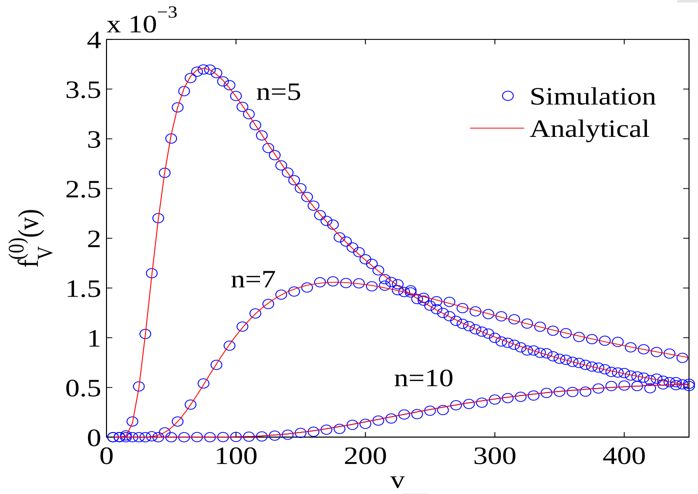

Figure 7: Comparison of simulated data points and the analytical p.d.f $f_V^{(0)}(v)$ (i.e., $\alpha = 0$) for different matrix dimensions with $\mu = 10$.

The above exact finite dimensional results pertaining to V have inherent algebraic complexity. For instance, obtaining an exact finite dimensional p.d.f. of V for a general value of α seems a formidable task. To circumvent this difficulty, it is natural to study the asymptotic behavior of the random variable V. The following corollary gives the asymptotic behavior of V for fixed α when m and n tend to ∞.

Corollary 8. *The scaled random variable V/n^3 converges in distribution to the random variable $1/X$ as m and n tend to ∞ with α fixed. More specifically, for fixed α, we have*

$$\lim_{n\to\infty} F_{V/n^3}(x) = F_{1/X}(x) = e^{-\frac{1}{x}} \det \left[I_{j-i}\left(\frac{2}{\sqrt{x}}\right)\right]_{i,j=1,\ldots,\alpha}. \tag{24}$$

Proof. We are aware that, as $m, n \to \infty$ with α fixed, $n\lambda_1$ converges in distribution to X and $\mathrm{tr}(\mathbf{W})/n^2$ converges in probability to 1[10]. Therefore, we can invoke the Slutsky's lemma (Van Der Vaart [61]) to establish the fact that

[10] The latter statement can easily be verified by considering the limiting m.g.f. of $\mathrm{tr}(\mathbf{W})/n^2$ as $n, m \to \infty$ with α fixed.

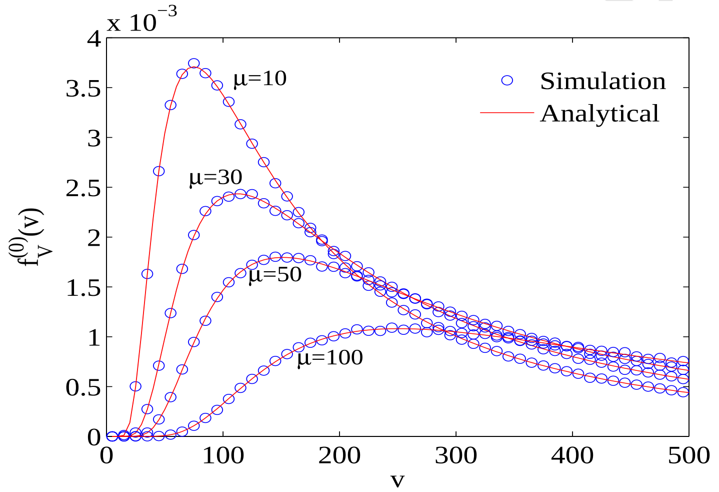

Figure 8: Illustration of the behavior of $f_V^{(0)}(v)$ for various values of μ with $n = 5$.

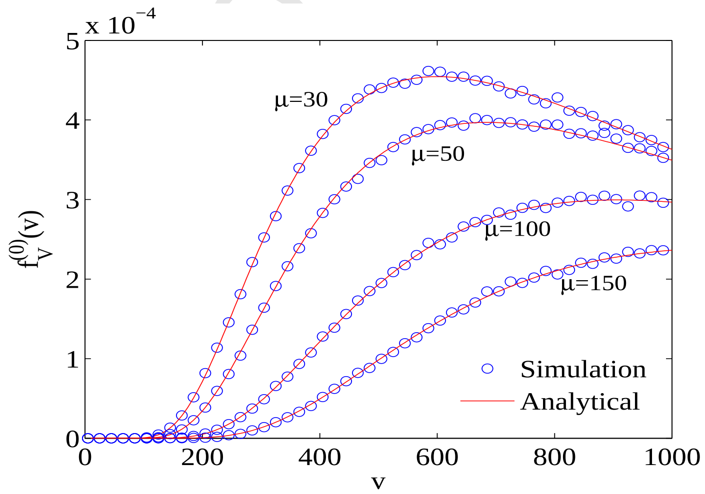

Figure 9: Illustration of the behavior of $f_V^{(0)}(v)$ for various values of μ with $n = 10$.

n^3/V converges in distribution to X. Now the final result follows by using the continuous mapping theorem and Corollary 5.

It is noteworthy that the limiting c.d.f. of V/n^3 does not depend on μ. Therefore, V/n^3 should have the same asymptotic limiting c.d.f. in the case corresponding to central Wishart matrices as well (see Akemann and Vivo [2, Eq. 4.34] for the result corresponding to central Wishart case).

5. The Average of the Reciprocal of a Certain Characteristic Polynomial

Here we consider the problem of determining the average of the reciprocal of a certain characteristic polynomial with respect to a complex non-central Wishart density with a rank-one mean. This particular problem corresponding to complex central Wishart matrices has been solved in Mehta [47], Fyodorov and Strahov [30]. A general framework to derive such averages based on duality relations has been proposed in Desrosiers [21]. However, the duality relation given in Desrosiers [21, Proposition 8] does not seem to apply here, since the stringent technical requirements for the validity of that formula are not satisfied by the parameters in our model of interest. Moreover, this particular case has not been addressed in a recent detailed analysis on the averages of characteristic polynomials by Forrester [27]. Therefore, in what follows, we derive the average of one of the basic forms of the reciprocal of the characteristic polynomial. The most general form, however, is not investigated here.

Let us consider the following average

$$\mathrm{E}_{\mathbf{W}}\left(\frac{1}{\det[z\mathbf{I}_n+\mathbf{W}]}\right)=\mathrm{E}_{\mathbf{\Lambda}}\left(\prod_{j=1}^{n}\frac{1}{z+\lambda_j}\right),\ |\arg z|<\pi, \tag{25}$$

the value of which is given in the following theorem.

Theorem 9. *Let* $\mathbf{W}\sim\mathcal{W}_n\left(m,\mathbf{I}_n,\mathbf{M}^*\mathbf{M}\right)$*, where* $\mathbf{M}$ *is rank-1 and* $\mathrm{tr}(\mathbf{M}^*\mathbf{M})=\mu$*. Then the average in (25) is given by*

$$\mathrm{E}_{\mathbf{W}}\left(\frac{1}{\det[z\mathbf{I}_n+\mathbf{W}]}\right)=z^{\alpha}\sum_{k=0}^{\infty}(-\mu)^k\Psi(k+n+\alpha;\alpha+1;z),\quad |\arg z|<\pi \tag{26}$$

where $\Psi(a;c;z)$ is the confluent hypergeometric function of the second kind.

Proof. See Appendix D.

Figures 10, 11 and 12 demonstrate the effect of each parameter (i.e., n, α and μ) on the average $\mathrm{E}_{\mathbf{W}}\left(1/\det[z\mathbf{I}_n + \mathbf{W}]\right)$. The agreement between the analytical and simulation results is clearly evident from the figure. This verifies the accuracy of our Theorem 9.

Remark 8. The behavior of $\mathrm{E}_{\mathbf{W}}\left(1/\det[z\mathbf{I}_n + \mathbf{W}]\right)$ for different K with the rank-one MIMO Rician model is expected to be inconsistent with that of fig. 12 due to (5) and (6).

6. Conclusions

Here we have addressed three problems related to the eigenvalues of a complex non-central Wishart matrix $\mathbf{W}$ with a rank-one mean. In particular, new expressions have been derived for the c.d.f. of the minimum eigenvalue, the p.d.f. of $\mathrm{tr}(\mathbf{W})/\lambda_1$, and the statistical average $\mathrm{E}_{\mathbf{W}}\left(1/\det[z\mathbf{I}_n + \mathbf{W}]\right)$. We have used a classical orthogonal polynomial based approach to derive the main results in this paper. One of the key advantages of this approach is that the dimensionality of the main results depends on $\alpha = m - n$. This α dependency is pivotal in deriving the microscopic limit of the minimum eigenvalue as well as establishing the stochastic convergence of the Demmel condition number. It turns out that, in the limit, the above two random quantities (once suitably scaled) do not depend on the rank-one mean, whereas the corresponding finite dimensional results depend on the rank-one mean through $\mu = \mathrm{tr}(\mathbf{M}^*\mathbf{M})$. This rank-one mean property also helps us express the average $\mathrm{E}_{\mathbf{W}}\left(1/\det[z\mathbf{I}_n + \mathbf{W}]\right)$ in terms of a single infinite summation.

7. Acknowledgments

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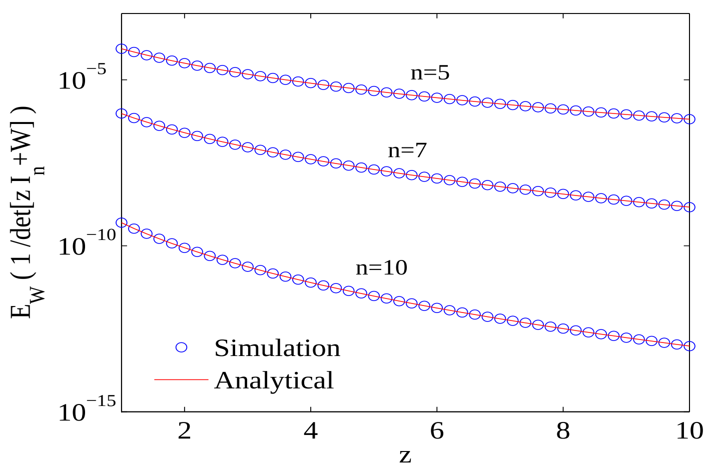

Figure 10: Comparison of simulated data points and the analytical average $\mathrm{E}_{\mathbf{W}}\left(1/\det[z\mathbf{I}_n + \mathbf{W}]\right)$ for different matrix dimensions with $\alpha = 2$ and $\mu = 10$.

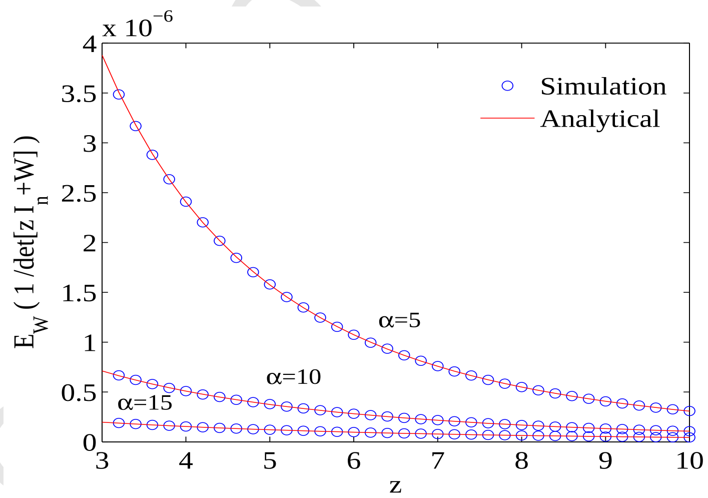

Figure 11: Illustration of the behavior of $\mathrm{E}_{\mathbf{W}}\left(1/\det[z\mathbf{I}_n + \mathbf{W}]\right)$ for various values of α with $n = 5$ and $\mu = 10$.

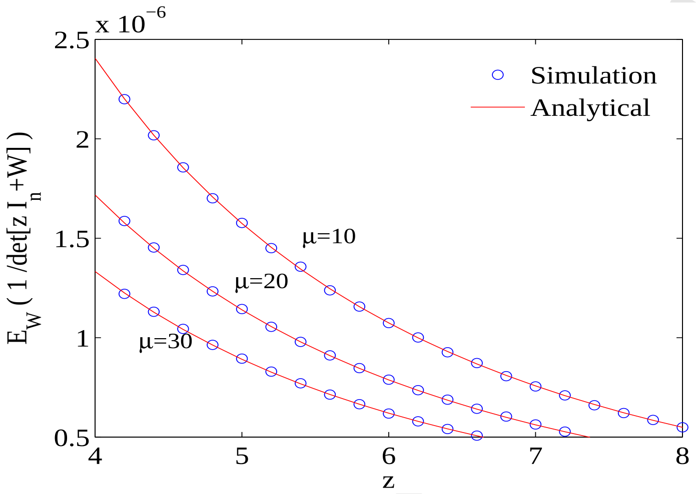

Figure 12: Illustration of the behavior of $\mathrm{E}_{\mathbf{W}}\left(1/\det[z\mathbf{I}_n+\mathbf{W}]\right)$ for various values of μ with $n=5$ and $\alpha=5$.

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Appendix A. Proof of Theorem 4

Since the joint p.d.f. is symmetric in $\lambda_1,\ldots,\lambda_n$, we can write (12) as

$$\Pr\left(\lambda_1 \geq x\right) = \frac{1}{n!}\int_{[0,\infty)^n} f_{\mathbf{\Lambda}}\left(\lambda_1+x,\ldots,\lambda_n+x\right)\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n.$$

Now it is convenient to use (3) to arrive at

$$\Pr\left(\lambda_1 \geq x\right) = \frac{\mathcal{K}_{n,\alpha}}{n!}\frac{e^{-\mu-nx}}{\mu^{n-1}}\sum_{k=1}^{n}\int_{[0,\infty)^n}\frac{{}_0F_1\left(\alpha+1;\mu(\lambda_k+x)\right)}{\prod_{\substack{i=1\\i\neq k}}^{n}\left(\lambda_k-\lambda_i\right)} \times\prod_{i=1}^{n}(\lambda_i+x)^{\alpha}e^{-\lambda_i}\Delta_n^2(\boldsymbol{\lambda})\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n. \quad \text{(A.1)}$$

Since each term in the above summation contributes the same amount, we may write (A.1) as

$$\Pr(\lambda_1 \geq x) = \frac{\mathcal{K}_{n,\alpha}}{(n-1)!}\frac{e^{-\mu-nx}}{\mu^{n-1}}\int_{[0,\infty)^n}\frac{{}_0F_1\left(\alpha+1;\mu(\lambda_1+x)\right)}{\prod_{i=2}^{n}\left(\lambda_1-\lambda_i\right)} \times \prod_{i=1}^{n}(\lambda_i+x)^{\alpha}e^{-\lambda_i}\Delta_n^2(\boldsymbol{\lambda})\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n.$$

Using the decomposition $\Delta_n^2(\boldsymbol{\lambda}) = \prod_{i=2}^{n}(\lambda_1-\lambda_i)^2\Delta_{n-1}^2(\boldsymbol{\lambda})$, we can rewrite the above multiple integral after relabeling the variables, $\lambda_1 = \lambda$, $\lambda_i = y_{i-1}, i = 2,\ldots,n$, as

$$\Pr(\lambda_1 \geq x) = \frac{\mathcal{K}_{n,\alpha}}{(n-1)!}\frac{e^{-\mu-nx}}{\mu^{n-1}}\int_{[0,\infty)}{}_0F_1\left(\alpha+1;\mu(\lambda+x)\right)(\lambda+x)^{\alpha}e^{-\lambda} \times (-1)^{(n-1)\alpha}Q_{n-1}\left(\lambda,-x,\alpha\right)\mathrm{d}\lambda \quad \text{(A.2)}$$

where

$$Q_n\left(a,b,\alpha\right) := \int_{[0,\infty)^n}\prod_{i=1}^{n}(a-y_i)(b-y_i)^{\alpha}e^{-y_i}\Delta_n^2(\boldsymbol{y})\mathrm{d}y_1\cdots\mathrm{d}y_n. \quad \text{(A.3)}$$

As shown in Appendix E, we can solve the above multiple integral in closed-form to yield

$$Q_n\left(a,b,\alpha\right) = \frac{\overline{\mathcal{K}}_{n,\alpha}}{(b-a)^{\alpha}}\det\left[L_{n+i-1}^{(0)}(a) \quad L_{n+i+1-j}^{(j-2)}(b)\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}} \quad \text{(A.4)}$$

where

$$\overline{\mathcal{K}}_{n,\alpha} = (-1)^{n+\alpha(n+\alpha)}\frac{\prod_{i=1}^{\alpha+1}(n+i-1)!\prod_{i=0}^{n-1}i!(i+1)!}{\prod_{i=1}^{\alpha-1}i!}.$$

Therefore, using (A.4) in (A.2) with some algebraic manipulation, we obtain

$$\Pr(\lambda_1 \geq x) = (-1)^{n+1}\frac{(n+\alpha-1)!}{\alpha!}\frac{e^{-\mu-nx}}{\mu^{n-1}}\det\left[\zeta_i(x) \quad L_{n+i-j}^{(j-2)}(-x)\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}} \quad \text{(A.5)}$$

where

$$\zeta_i(x) = \int_0^{\infty}{}_0F_1\left(\alpha+1;\mu(\lambda+x)\right)e^{-\lambda}L_{n+i-2}^{(0)}(\lambda)\mathrm{d}\lambda.$$

The remaining task is to evaluate the above integral, which does not seem to have a simple closed-form solution. Therefore, we expand the hypergeometric function with its equivalent power series and use Corollary 3 to arrive at

$$\begin{aligned}\zeta_i(x) &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{\mu^{l+k}x^k}{k!l!(\alpha+1)_{l+k}}\int_0^{\infty}\lambda^l e^{-\lambda}L_{n+i-2}^{(0)}(\lambda)\mathrm{d}\lambda\\ &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{\mu^{l+k}x^k}{k!(\alpha+1)_{l+k}}\frac{(-l)_{n+i-2}}{(1)_{n+i-2}}.\end{aligned} \tag{A.6}$$

Following (10), we observe that the quantity $(-l)_{n+i-2}$ is non-zero only when $l \geq n+i-2$. Therefore, we shift the summation index l with some algebraic manipulation to yield

$$\begin{aligned}\zeta_i(x) = \frac{(-1)^{n+i}\mu^{n+i-2}\alpha!}{(n+i+\alpha-2)!}&\sum_{k=0}^{\infty}\frac{(x\mu)^k}{k!(n+i+\alpha-1)_k}\\ &\times {}_1F_1(n+i-1; n+\alpha+i+k-1;\mu)\end{aligned} \tag{A.7}$$

where we have used the relation

$$(\alpha+1)_{k+i+n+l-2} = \frac{(\alpha+i+n-2)!}{\alpha!}(\alpha+i+n+k-1)_l(\alpha+i+n-1)_k.$$

Substituting (A.7) back into (A.5) with some algebra then gives

$$\Pr(\lambda_1 \geq x) = (n+\alpha-1)!e^{-nx}\det\left[(-\mu)^{i-1}\psi_i(\mu,x) \quad L_{n+i-j}^{(j-2)}(-x)\right]_{\substack{i=1,\dots,\alpha+1\\ j=2,\dots,\alpha+1}} \tag{A.8}$$

where we have used the Kummer relation (Erdélyi [25]), ${}_1F_1(a;c;z) = e^z{}_1F_1(c-a;c,-z)$. Finally, using (A.8) in (11) gives the c.d.f. of the minimum eigenvalue, which concludes the proof.

Appendix B. Proof of Corollary 5

It is convenient to start with (A.5) which we rewrite with the help of (A.6) as

$$\begin{aligned}\Pr(\lambda_1 \geq x) = (-1)^{n+1}\frac{e^{-\mu-nx}}{\alpha!\mu^{n-1}}&\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\mu^{k+l}x^l(-k)_{n-1}}{l!(\alpha+1)_{k+l}}\\ &\times\det\left[\frac{(n)_\alpha}{(n)_{i-1}}\frac{(-k)_{n+i-2}}{(-k)_{n-1}} \quad L_{n+i-j}^{(j-2)}(-x)\right]_{\substack{i=1,\dots,\alpha+1\\ j=2,\dots,\alpha+1}}.\end{aligned}$$

We may further rewrite it as

$$\Pr(\lambda_1 \geq x) = \frac{e^{-\mu - nx}}{\alpha! \mu^{n-1}} \sum_{k=n-1}^{\infty} \sum_{l=0}^{\infty} \frac{\mu^{k+l} x^l \, k!}{l! (\alpha+1)_{k+l} (k-n+1)!} \\ \times \det \left[\frac{(n)_\alpha}{(n)_{i-1}} \frac{(-k)_{n+i-2}}{(-k)_{n-1}} \quad L_{n+i-j}^{(j-2)}(-x) \right]_{\substack{i=1,\dots,\alpha+1 \\ j=2,\dots,\alpha+1}} . \tag{B.1}$$

Now we use the decomposition

$$\frac{(-k)_{n+i-2}}{(-k)_{n-1}} = (-1)^{i-1} \prod_{p=0}^{i-2} (k-n+1-p) \tag{B.2}$$

to obtain

$$\Pr(\lambda_1 \geq x) = \frac{e^{-\mu - nx}}{\alpha! \mu^{n-1}} \sum_{k=n-1}^{\infty} \sum_{l=0}^{\infty} \frac{\mu^{k+l} x^l k!}{l! (\alpha+1)_{k+l} (k-n+1)!} \\ \times \det \left[(-1)^{i-1} \frac{(n)_\alpha}{(n)_{i-1}} \prod_{p=0}^{i-2} (k-n+1-p) \right. \\ \left. L_{n+i-j}^{(j-2)}(-x) \right]_{\substack{i=1,\dots,\alpha+1 \\ j=2,\dots,\alpha+1}} . \tag{B.3}$$

Since the first infinite summation begins with $k = n-1$, we may reorganize the sum with respect to k to yield

$$\Pr(\lambda_1 \geq x) = (n)_\alpha e^{-\mu - nx} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\mu^{k+l} x^l (n)_k}{k! l! (n)_{\alpha+l+k}} \\ \times \det \left[\frac{(-1)^{i-1}}{(n)_{i-1}} \prod_{p=0}^{i-2} (k-p) \quad L_{n+i-j}^{(j-2)}(-x) \right]_{\substack{i=1,\dots,\alpha+1 \\ j=2,\dots,\alpha+1}} . \tag{B.4}$$

At this juncture, we focus on further simplification of the columns involving the generalized Laguerre polynomials. To this end, we use (7) to obtain

$$\begin{aligned}
&\Pr\left(\lambda_1 \geq x\right)\\
&= \frac{(n)_\alpha e^{-\mu-nx}}{\prod_{j=1}^{\alpha}(j-1)!} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\mu^{k+l}x^l(n)_k}{k!l!(n)_{\alpha+l+k}}\\
&\quad \times \det\left[\frac{(-1)^{i-1}}{(n)_{i-1}}\prod_{p=0}^{i-2}(k-p)\right.\\
&\qquad\qquad \left.\frac{(n+i-2)!}{(n+i-j)!}\sum_{k_j=0}^{n+i-j}\frac{(-n-i+j)_{k_j}}{(j-1)_{k_j}}\frac{(-x)^{k_j}}{(k_j)!}\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}}. \qquad \text{(B.5)}
\end{aligned}$$

Further manipulation in this form is highly undesirable due to the i, j dependent summations upper limits in the $2, \ldots, \alpha+1$ columns of the determinant. To circumvent this problem, we use the factorization

$$\begin{aligned}
(-n-i+j)_{k_j} &= (-n-i+j)_{k_j}\frac{(-n-\alpha-1+j)_{k_j}}{(-n-\alpha-1+j)_{k_j}}\\
&= \frac{(n+i-j)!}{(n+\alpha+1-j)!}(-n-\alpha-1+j)_{k_j}\prod_{p=0}^{\alpha-i}(\overline{c}_j-p), \qquad \text{(B.6)}
\end{aligned}$$

where $\overline{c}_j = n+\alpha+1-j-k_j$, in (B.5) with some algebraic manipulation to yield

$$\begin{aligned}
&\Pr\left(\lambda_1 \geq x\right)\\
&= \frac{(n)_\alpha\, e^{-\mu-nx}}{\prod_{j=1}^{\alpha}(j-1)!(n+\alpha-j)!} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\mu^{k+l}x^l(n)_k}{k!l!(n)_{\alpha+l+k}}\\
&\quad \times \det\left[\frac{(-1)^{i-1}}{(n)_{i-1}}\prod_{p=0}^{i-2}(k-p)\right.\\
&\qquad\qquad \left.(n+i-2)!\sum_{k_j=0}^{n+\alpha+1-j}\frac{(-n-\alpha-1+j)_{k_j}}{(j-1)_{k_j}}\frac{(-x)^{k_j}}{(k_j)!}\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}} \\
&\hfill \text{(B.7)}
\end{aligned}$$

from which we obtain

$$\begin{aligned}\Pr(\lambda_1 \geq x) = \frac{(n)_\alpha \, e^{-\mu - nx}}{\prod_{j=1}^{\alpha}(j-1)!} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\mu^{k+l} x^l (n)_k}{k! l! (n)_{\alpha+l+k}} \\ \times \sum_{k_1=0}^{n+\alpha-1}\sum_{k_2=0}^{n+\alpha-2}\cdots\sum_{k_\alpha=0}^{n}\prod_{j=1}^{\alpha}\frac{(-n-\alpha+j)_{k_j}}{(j)_{k_j}}\frac{(-x)^{k_j}}{(k_j)!} \\ \times \det\left[(-1)^i \prod_{p=0}^{i-1}\frac{(k-p)}{(n+p)^2} \prod_{p=0}^{\alpha-i-1}(\tilde{c}_j - p)\right]_{\substack{i=0,\ldots,\alpha \\ j=1,\ldots,\alpha}}\end{aligned} \tag{B.8}$$

where $\tilde{c}_j = n+\alpha-j-k_j$ and in the second equality, we have translated the indices $i = 1, \ldots, \alpha+1$ to $i = 0, \ldots, \alpha$ and $j = 2, \ldots, \alpha+1$ to $j = 1, \ldots, \alpha$. Note that, under the current index assignment, an empty product is interpreted as unity. Keeping in mind that we are interested in the limit as $n \to \infty$, we may further simplify the above determinant with the help of Dharmawansa et al. [24, Lemma A.1] to yield

$$\begin{aligned}\Pr(\lambda_1 \geq x) = \frac{(n)_\alpha \, e^{-\mu - nx}}{\prod_{j=1}^{\alpha}(j-1)!} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\mu^{k+l} x^l (n)_k}{k! l! (n)_{\alpha+l+k}} \\ \times \sum_{k_1=0}^{n+\alpha-1}\sum_{k_2=0}^{n+\alpha-2}\cdots\sum_{k_\alpha=0}^{n}\prod_{j=1}^{\alpha}\frac{(-n-\alpha+j)_{k_j}}{(j)_{k_j}}\frac{(-x)^{k_j}}{(k_j)!}\Xi(n,k,\tilde{\mathbf{c}})\end{aligned} \tag{B.9}$$

where

$$\Xi(n,k,\tilde{\mathbf{c}}) = \begin{vmatrix} 1+\mathrm{o}(1) & \tilde{c}_1^{\alpha} & \tilde{c}_2^{\alpha} & \cdots & \tilde{c}_\alpha^{\alpha} \\ \frac{-k}{n^2} + \mathrm{o}\left(\frac{1}{n^2}\right) & \tilde{c}_1^{\alpha-1} & \tilde{c}_2^{\alpha-1} & \cdots & \tilde{c}_\alpha^{\alpha-1} \\ \frac{k(k-1)}{n^2(n+1)^2} + \mathrm{o}\left(\frac{1}{n^4}\right) & \tilde{c}_1^{\alpha-2} & \tilde{c}_2^{\alpha-2} & \cdots & \tilde{c}_\alpha^{\alpha-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{\alpha-2}\prod_{p=0}^{\alpha-3}\frac{(k-p)}{(n+p)^2} + \mathrm{o}\left(\frac{1}{n^{2(\alpha-2)}}\right) & \tilde{c}_1^{2} & \tilde{c}_2^{2} & \cdots & \tilde{c}_\alpha^{2} \\ (-1)^{\alpha-1}\prod_{p=0}^{\alpha-2}\frac{(k-p)}{(n+p)^2} & \tilde{c}_1 & \tilde{c}_2 & \cdots & \tilde{c}_\alpha \\ (-1)^{\alpha}\prod_{p=0}^{\alpha-1}\frac{(k-p)}{(n+p)^2} & 1 & 1 & \cdots & 1 \end{vmatrix} \tag{B.10}$$

with $\tilde{\mathbf{c}} = (\tilde{c}_1, \ldots, \tilde{c}_\alpha)$ and for non-zero $g(x)$, $f(x) = \mathrm{o}(g(x))$ is equivalent to $\lim_{x\to\infty} f(x)/g(x) = 0$.

Now let us consider the expression $\Pr(\lambda_1 \geq x/n)$, which can be written as

$$\Pr\left(\lambda_1 \geq \frac{x}{n}\right) = \frac{(n)_\alpha\, e^{-\mu-x}}{\prod_{j=1}^{\alpha}(j-1)!} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\mu^{k+l}x^l(n)_k}{k!l!(n)_{\alpha+l+k}n^l} \times \sum_{k_1=0}^{n+\alpha-1}\sum_{k_2=0}^{n+\alpha-2}\cdots\sum_{k_\alpha=0}^{n}\prod_{j=1}^{\alpha}\frac{(-n-\alpha+j)_{k_j}}{(j)_{k_j}}\frac{(-\frac{x}{n})^{k_j}}{(k_j)!}\Xi(n,k,\tilde{\mathbf{c}}). \tag{B.11}$$

Assuming that we have the freedom to take the limits term by term and observing that only the terms corresponding to $l=0$ contribute to a non-zero limit as $n\to\infty$, we take the limit of (B.11) as $n\to\infty$ to yield

$$\lim_{n\to\infty}\Pr\left(\lambda_1 \geq \frac{x}{n}\right) = \frac{e^{-\mu-x}}{\prod_{j=1}^{\alpha}(j-1)!}\sum_{k=0}^{\infty}\frac{\mu^k}{k!}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_\alpha=0}^{\infty}\prod_{j=1}^{\alpha}\frac{x^{k_j}}{(j)_{k_j}k_j!}\lim_{n\to\infty}\Xi(n,k,\tilde{\mathbf{c}}). \tag{B.12}$$

We now focus on the limiting value of the remaining determinant $\Xi(n,k,\tilde{\mathbf{c}})$. Since each of the terms $\tilde{c}_j,\ j=1,\ldots,\alpha$, contains n, a straightforward term-by-term limit will give an indeterminate form. To circumvent this problem, we try to simplify the determinant as much as possible before taking the limits. To this end, we expand the determinant using its first column as

$$\Xi(n,k,\tilde{\mathbf{c}}) = \sum_{i=1}^{\alpha+1}(-1)^{i+1}\Xi_{1,i}(n,k,\tilde{\mathbf{c}})\mathrm{M}_{1,i} \tag{B.13}$$

where $\mathrm{M}_{1,i}$ is the minor corresponding to the ith element of the first column. As such, we can represent the $\mathrm{M}_{1,i}$ as the determinant of an $\alpha\times\alpha$ matrix as

$$\mathrm{M}_{1,i} = \begin{vmatrix} \tilde{c}_1^{\alpha} & \tilde{c}_2^{\alpha} & \cdots & \tilde{c}_\alpha^{\alpha} \\ \tilde{c}_1^{\alpha-1} & \tilde{c}_2^{\alpha-1} & \cdots & \tilde{c}_\alpha^{\alpha-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_1^{\alpha-i+2} & \tilde{c}_2^{\alpha-i+2} & \cdots & \tilde{c}_\alpha^{\alpha-i+2} \\ \tilde{c}_1^{\alpha-i} & \tilde{c}_2^{\alpha-i} & \cdots & \tilde{c}_\alpha^{\alpha-i} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}. \tag{B.14}$$

A few observations on the evaluation of this determinant are in order now. Clearly, by invoking the factor theorem, it can be easily seen that there exists $\alpha(\alpha-1)/2$ factors of the form $\tilde{c}_i - \tilde{c}_j$ with $1 \leq i < j \leq \alpha$. Most importantly, each of such factors is free of n (i.e., because $\tilde{c}_i - \tilde{c}_j = j + k_j - i - k_i$). Since the original minor consists of monomials of degree ν_1 given by

$$\nu_1 = 1 + \cdots + \alpha - (\alpha - i + 1) = \frac{1}{2}(\alpha^2 - \alpha + 2i - 2), \tag{B.15}$$

the other factor term should be of degree ν_2, which is given by

$$\nu_2 = \nu_1 - \frac{1}{2}\alpha(\alpha-1) = i - 1. \tag{B.16}$$

Therefore, we can write

$$\begin{aligned} \mathrm{M}_{1,i} &= \left(\prod_{1 \leq i < j \leq \alpha} (j + k_j - i - k_i) \right) \left(\sum_{k=0}^{i-1} a_{k,i} n^k \right) \\ &= \Delta_\alpha(\mathbf{c}) \left(\sum_{k=0}^{i-1} a_{k,i} n^k \right) \end{aligned} \tag{B.17}$$

where $\mathbf{c} = (c_1, \ldots, c_\alpha)$ with $c_l = l + k_l$ and $a_{k,i},\ k = 1, \ldots, i-1$, are constant coefficients free of n. This in turn gives

$$\Xi(n, k, \tilde{\mathbf{c}}) = \Delta_\alpha(\mathbf{c}) \sum_{i=1}^{\alpha+1} (-1)^{i+1} \Xi_{1,i}(n, k, \tilde{\mathbf{c}}) \sum_{k=0}^{i-1} a_{k,i} n^k. \tag{B.18}$$

Since, for large n, each of $\Xi_{1,i}(n, k, \tilde{\mathbf{c}})$ (i.e., the ith element of the first column of $\Xi(n, k, \tilde{\mathbf{c}})$) has the lowest order term of $1/n^{2(i-1)}$, one can easily obtain

$$\begin{aligned} \lim_{n\to\infty} \Xi(n, k, \tilde{\mathbf{c}}) &= \Delta_\alpha(\mathbf{c}) \sum_{i=1}^{\alpha+1} (-1)^{i+1} \lim_{n\to\infty} \left(\Xi_{1,i}(n, k, \tilde{\mathbf{c}}) \sum_{k=0}^{i-1} a_{k,i} n^k \right) \\ &= \Delta_\alpha(\mathbf{c}) \end{aligned} \tag{B.19}$$

where we have used the fact that $a_{0,1} = 1$. Thus, we can write (B.12) as

$$\lim_{n\to\infty} \Pr\left(\lambda_1 \geq \frac{x}{n} \right) = \frac{e^{-x}}{\prod_{j=1}^{\alpha} (j-1)!} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_\alpha=0}^{\infty} \prod_{j=1}^{\alpha} \frac{x^{k_j}}{(j)_{k_j} k_j!} \Delta_\alpha(\mathbf{c}) \tag{B.20}$$

from which we obtain

$$\lim_{n\to\infty} \Pr\left(\lambda_1 \geq \frac{x}{n}\right) = \frac{e^{-x}}{\prod_{j=1}^{\alpha}(j-1)!} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_\alpha=0}^{\infty}\prod_{j=1}^{\alpha}\frac{x^{k_j}}{(j)_{k_j}k_j!} \det\left[(k_j+j)^{i-1}\right]_{i,j=1,\ldots,\alpha}. \tag{B.21}$$

To facilitate further manipulations, keeping in mind that Vandermonde determinant is invariant to constant shift of its arguments, we rewrite the above equation as

$$\lim_{n\to\infty} \Pr\left(\lambda_1 \geq \frac{x}{n}\right) = \frac{e^{-x}}{\prod_{j=1}^{\alpha}(j-1)!} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_\alpha=0}^{\infty}\prod_{j=1}^{\alpha}\frac{x^{k_j}}{(j)_{k_j}k_j!} \det\left[(k_j+j+\beta-1)^{i-1}\right]_{i,j=1,\ldots,\alpha} \tag{B.22}$$

where β is an arbitrary non-negative number. In view of Dharmawansa et al. [24, Lemma A.1], the alternative representation of the Vandermonde determinant gives

$$\begin{aligned}
\lim_{n\to\infty} \Pr\left(\lambda_1 \geq \frac{x}{n}\right)
&= \frac{e^{-x}}{\prod_{j=1}^{\alpha}(j-1)!} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_\alpha=0}^{\infty}\prod_{j=1}^{\alpha}\frac{x^{k_j}}{(j)_{k_j}k_j!} \\
&\qquad \times \det\left[\prod_{p=0}^{i-2}(k_j+j+\beta-1-p)\right]_{i,j=1,\ldots,\alpha} \\
&= \frac{e^{-x}}{\prod_{j=1}^{\alpha}(j-1)!} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_\alpha=0}^{\infty}\prod_{j=1}^{\alpha}\frac{x^{k_j}}{(j)_{k_j}k_j!} \det\left[\frac{(k_j+j+\beta-1)!}{(k_j+j+\beta-i)!}\right]_{i,j=1,\ldots,\alpha} \\
&= e^{-x}\prod_{j=1}^{\alpha}\frac{(\beta+j-1)!}{(j-1)!} \det\left[\frac{1}{(1)_{j+\beta-i}}\sum_{k_j=0}^{\infty}\frac{(\beta+j)_{k_j}}{(j)_{k_j}(j+\beta+1-i)_{k_j}}\frac{x^{k_j}}{k_j!}\right]_{i,j=1,\ldots,\alpha}.
\end{aligned} \tag{B.23}$$

Since β is an arbitrary non-negative number, we may set it to zero and use the relation

$$z^{-\frac{\nu}{2}} I_\nu(2\sqrt{z}) = \frac{1}{(1)_\nu}\sum_{k=0}^{\infty}\frac{z^k}{k!(\nu+1)_k} \tag{B.24}$$

with $I_\nu(z)$ denoting the modified Bessel function of the first kind and order ν, to arrive at

$$\lim_{n\to\infty} \Pr\left(\lambda_1 \geq \frac{x}{n}\right) = e^{-x} \det\left[x^{\frac{i-j}{2}} I_{j-i}(2\sqrt{x})\right]_{i,j=1,\ldots,\alpha}$$
$$= e^{-x} \det\left[I_{j-i}(2\sqrt{x})\right]_{i,j=1,\ldots,\alpha}. \tag{B.25}$$

330 Finally, (B.25) along with the fact that $\lim_{n\to\infty} F_{n\lambda_1}(x) = 1 - \lim_{n\to\infty} \Pr\left(n\lambda_1 \geq x\right) = F_X(x)$ concludes the proof.

Appendix C. Proof of Theorem 6

By definition, the m.g.f. of V can be written as

$$\mathfrak{M}_V(s) = \mathrm{E}_{\boldsymbol{\Lambda}}\left(e^{-s\frac{\sum_{j=1}^n \lambda_j}{\lambda_1}}\right), \quad \Re(s) \geq 0,$$

which has the following multiple integral representation

$$\mathfrak{M}_V(s) = e^{-s} \int_{0\leq\lambda_1\leq\cdots\leq\lambda_n<\infty} e^{-s\frac{\sum_{j=2}^n \lambda_j}{\lambda_1}} f_{\boldsymbol{\Lambda}}(\lambda_1,\ldots,\lambda_n) \mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n.$$

Since the argument of the exponential function is symmetric in $\lambda_2,\ldots,\lambda_n$, it is convenient to introduce the substitution $\lambda_1 = x$ and rewrite the multiple integral, keeping the integration with respect to x last, as

$$\mathfrak{M}_V(s) = e^{-s} \int_0^\infty \int_{x\leq\lambda_2\leq\ldots\leq\lambda_n<\infty} e^{-s\frac{\sum_{j=2}^n \lambda_j}{x}} f_{\boldsymbol{\Lambda}}(x,\lambda_2,\ldots,\lambda_n) \mathrm{d}\lambda_2\cdots\mathrm{d}\lambda_n \mathrm{d}x. \tag{C.1}$$

To be consistent with the above setting, we may restructure the joint p.d.f. of $\boldsymbol{\Lambda}$ given in (3) as

$$f_{\boldsymbol{\Lambda}}(x,\lambda_2,\ldots,\lambda_n) = \mathcal{K}_{n,\alpha} \frac{e^{-\mu}}{\mu^{n-1}} x^\alpha e^{-x} \prod_{i=2}^n \lambda_i^\alpha e^{-\lambda_i}(x-\lambda_i)^2 \Delta_{n-1}^2(\boldsymbol{\lambda})$$
$$\times \left(\frac{{}_0F_1\left(\alpha+1;\mu x\right)}{\prod_{i=2}^n (x-\lambda_i)} + \sum_{k=2}^n \frac{{}_0F_1\left(\alpha+1;\mu\lambda_k\right)}{(\lambda_k - x)\prod_{\substack{i=2\\i\neq k}}^n (\lambda_k-\lambda_i)}\right) \tag{C.2}$$

where we have used the decomposition $\Delta_n^2(\boldsymbol{\lambda}) = (x-\lambda_i)^2\Delta_{n-1}^2(\boldsymbol{\lambda})$. Now we use (C.2) in (C.1) with some algebraic manipulation to obtain

$$\mathfrak{M}_V(s) = \mathfrak{P}(s) + \mathfrak{S}(s) \tag{C.3}$$

where

$$\mathfrak{P}(s) = \mathcal{K}_{n,\alpha}\frac{e^{-\mu-s}}{\mu^{n-1}}\int_0^\infty e^{-x}x^\alpha {}_0F_1\left(\alpha+1;\mu x\right) \times \left(\int_{x\leq\lambda_2\leq\cdots\leq\lambda_n<\infty}\prod_{i=2}^n e^{-\left(1+\frac{s}{x}\right)\lambda_i}\lambda_i^\alpha(x-\lambda_i)\Delta_{n-1}^2(\boldsymbol{\lambda})\mathrm{d}\lambda_2\cdots\mathrm{d}\lambda_n\right)\mathrm{d}x \tag{C.4}$$

and

$$\mathfrak{S}(s) = \mathcal{K}_{n,\alpha}\frac{e^{-\mu-s}}{\mu^{n-1}}\int_0^\infty e^{-x}x^\alpha\left(\int_{x\leq\lambda_2\leq\cdots\leq\lambda_n<\infty}\sum_{k=2}^n\frac{{}_0F_1\left(\alpha+1;\mu\lambda_k\right)}{(\lambda_k-x)\prod\limits_{\substack{i=2\\i\neq k}}^n(\lambda_k-\lambda_i)} \times\prod_{i=2}^n\lambda_i^\alpha e^{-\lambda_i}(x-\lambda_i)^2\Delta_{n-1}^2(\boldsymbol{\lambda})\mathrm{d}\lambda_2\cdots\mathrm{d}\lambda_n\right)\mathrm{d}x. \tag{C.5}$$

The remainder of this proof focuses on evaluating the above two multiple integrals. Since the two integrals do not share a common structure, in what follows, we will evaluate them separately.

Let us begin with (C.4). Clearly, the inner multiple integral is symmetric in $\lambda_2,\ldots,\lambda_n$. Thus, we can remove the ordered region of integration to yield

$$\mathfrak{P}(s) = \frac{\mathcal{K}_{n,\alpha}}{(n-1)!}\frac{e^{-\mu-s}}{\mu^{n-1}}\int_0^\infty e^{-x}x^\alpha {}_0F_1\left(\alpha+1;\mu x\right) \times\left(\int_{[x,\infty)^{n-1}}\prod_{i=2}^n e^{-\left(1+\frac{s}{x}\right)\lambda_i}\lambda_i^\alpha(x-\lambda_i)\Delta_{n-1}^2(\boldsymbol{\lambda})\mathrm{d}\lambda_2\cdots\mathrm{d}\lambda_n\right)\mathrm{d}x.$$

Now we apply the change of variables $y_{i-1}=(x+s)(\lambda_i-x)/x,\ i=2,\ldots,n$, to the inner $(n-1)$ fold integral with some algebraic manipulation to obtain

$$\mathfrak{P}(s) = (-1)^{(n-1)(1+\alpha)}\frac{\mathcal{K}_{n,\alpha}}{(n-1)!}\frac{e^{-\mu-sn}}{\mu^{n-1}}\int_0^\infty e^{-nx}x^{n(n-1+\alpha)}\frac{{}_0F_1\left(\alpha+1;\mu x\right)}{(x+s)^{(n+\alpha)(n-1)}} \times R_{n-1}(-(s+x),\alpha)\mathrm{d}x$$

where

$$R_n(a,\alpha) = \int_{[0,\infty)^n} \prod_{i=1}^{n} e^{-y_j} y_j (a-y_j)^{\alpha} \Delta_n^2(\mathbf{y}) \mathrm{d}y_1 \cdots \mathrm{d}y_n.$$

Following Mehta [47, Section 22.2.2], we can solve the above integral to yield[11]

$$R_n(a,\alpha) = (-1)^{n\alpha} \prod_{j=0}^{n-1} (j+1)!(j+1)! \prod_{j=0}^{\alpha-1} \frac{(n+j)!}{j!} \det\left[L_{n+i-j}^{(j)}(a)\right]_{i,j=1,\dots,\alpha}. \tag{C.6}$$

Therefore, we obtain

$$\begin{aligned}\mathfrak{P}(s) = (-1)^{(n-1)} \frac{(n-1)!}{\alpha!} \frac{e^{-\mu-sn}}{\mu^{n-1}} \\ \times \int_0^\infty e^{-nx} x^{n(n-1+\alpha)} \frac{{}_0F_1\left(\alpha+1;\mu x\right)}{(x+s)^{(n+\alpha)(n-1)}} \\ \times \det\left[L_{n+i-j-1}^{(j)}(-x-s)\right]_{i,j=1,\dots,\alpha} \mathrm{d}x.\end{aligned} \tag{C.7}$$

Although further manipulation in this form is feasible, it is convenient to leave the solution in the current form. Next we focus on solving the multiple integral given in (C.5).

By symmetry, we convert the ordered region of integration in (C.5) to an unordered region to yield

$$\begin{aligned}\mathfrak{S}(s) = \frac{\mathcal{K}_{n,\alpha}}{(n-1)!} \frac{e^{-\mu-s}}{\mu^{n-1}} \int_0^\infty e^{-x} x^{\alpha} \Bigg(\int_{[x,\infty)^{n-1}} \sum_{k=2}^{n} \frac{{}_0F_1\left(\alpha+1;\mu\lambda_k\right)}{(\lambda_k - x)\prod_{\substack{i=2\\i\neq k}}^{n}(\lambda_k-\lambda_i)} \\ \times \prod_{i=2}^{n} \lambda_i^{\alpha} e^{-\lambda_i} (x-\lambda_i)^2 \Delta_{n-1}^2(\boldsymbol{\lambda}) \mathrm{d}\lambda_2 \cdots \mathrm{d}\lambda_n \Bigg) \mathrm{d}x.\end{aligned}$$

Since each term in the above summation contributes the same amount, we can

[11] Specific steps pertaining to this evaluation are not given here as the detailed steps of solving a similar integral have been given in Dharmawansa et al. [24].

further simplify the multiple integral giving

$$\mathfrak{S}(s) = \frac{\mathcal{K}_{n,\alpha}}{(n-2)!}\frac{e^{-\mu-s}}{\mu^{n-1}}\int_0^\infty e^{-x}x^\alpha\left(\int_{[x,\infty)^{n-1}} \frac{{}_0F_1\left(\alpha+1;\mu\lambda_2\right)}{(\lambda_2-x)\prod_{i=3}^{n}(\lambda_2-\lambda_i)}\right.$$
$$\left.\times\prod_{i=2}^{n}\lambda_i^\alpha e^{-\lambda_i}(x-\lambda_i)^2\Delta_{n-1}^2(\boldsymbol{\lambda})\mathrm{d}\lambda_2\cdots\mathrm{d}\lambda_n\right)\mathrm{d}x,$$

from which we obtain, after using the decomposition $\Delta_{n-1}^2(\boldsymbol{\lambda}) = \prod_{j=3}^{n}(\lambda_2 - \lambda_j)^2\Delta_{n-2}^2(\boldsymbol{\lambda})$,

$$\begin{aligned}&\mathfrak{S}(s)\\ &= \frac{\mathcal{K}_{n,\alpha}}{(n-2)!}\frac{e^{-\mu-s}}{\mu^{n-1}}\int_0^\infty e^{-x}x^\alpha\left\{\int_x^\infty \lambda_2^\alpha(\lambda_2-x){}_0F_1\left(\alpha+1;\mu\lambda_2\right)e^{-\left(1+\frac{s}{x}\right)\lambda_2}\right.\\ &\left.\times\left(\int_{[x,\infty)^{n-2}}\prod_{i=3}^{n}\lambda_i^\alpha e^{-\left(1+\frac{s}{x}\right)\lambda_i}(\lambda_2-\lambda_i)(x-\lambda_i)^2\Delta_{n-2}^2(\boldsymbol{\lambda})\mathrm{d}\lambda_3\cdots\mathrm{d}\lambda_n\right)\mathrm{d}\lambda_2\right\}\mathrm{d}x.\end{aligned}$$

Now we apply the variable transformations, $y = \lambda_2 - x,\ y_{i-2} = (x+s)(\lambda_i - x)/x,\ \ i = 3,\ldots,n$, in the above multiple integral to yield

$$\begin{aligned}&\mathfrak{S}(s)\\ &= (-1)^{n\alpha}\frac{\mathcal{K}_{n,\alpha}}{(n-2)!}\frac{e^{-\mu-sn}}{\mu^{n-1}}\int_0^\infty\frac{e^{-xn}x^\alpha}{\left(1+\frac{s}{x}\right)^{(n-2)(n+\alpha+1)}}\left\{\int_0^\infty y(y+x)^\alpha e^{-\left(1+\frac{s}{x}\right)y}\right.\\ &\left.\times\,{}_0F_1\left(\alpha+1;\mu(y+x)\right)T_{n-2}\left(y\left(1+\frac{s}{x}\right),-s-x,\alpha\right)\mathrm{d}y\right\}\mathrm{d}x\end{aligned}$$

where

$$T_n(a,b,\alpha) := \int_{[0,\infty)^n}\prod_{i=1}^{n}(a-y_i)(b-y_i)^\alpha e^{-y_i}y_i^2\Delta_n^2(\mathbf{y})\mathrm{d}y_1\cdots\mathrm{d}y_n. \tag{C.8}$$

It is not difficult to observe that $T_n(a,b,\alpha)$ and $Q_n(a,b,\alpha)$ defined in (A.3) share a common structure up to a certain Laguerre weight. Therefore, we can readily follow similar arguments as shown in Appendix E with the modified monic orthogonal polynomials given by $\mathsf{P}_k(x) = (-1)^k k! L_k^{(2)}(x)$ to arrive at

$$T_n(a,b,\alpha) := \frac{(-1)^{n+\alpha(n+\alpha)}\widetilde{\mathcal{K}}_{n,\alpha}}{(b-a)^\alpha}\det\left[L_{n+i-1}^{(2)}(a)\quad L_{n+i+1-j}^{(j)}(b)\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}}$$

where

$$\widetilde{\mathcal{K}}_{n,\alpha} = \frac{\prod_{j=1}^{\alpha+1}(n+j-1)!\prod_{j=0}^{n-1}(j+1)!(j+2)!}{\prod_{j=0}^{\alpha-1} j!}.$$

This in turn gives

$$\begin{aligned}
&\mathfrak{S}(s)\\
&= (-1)^n \frac{(n-1)!}{\alpha!}\frac{e^{-\mu-sn}}{\mu^{n-1}}\\
&\quad\times \int_0^\infty \frac{e^{-xn}x^\alpha}{\left(1+\frac{s}{x}\right)^{(n-1)(n+\alpha)-2}} \left\{\int_0^\infty ye^{-\left(1+\frac{s}{x}\right)y} {}_0F_1\left(\alpha+1;\mu(y+x)\right)\right.\\
&\quad\left.\times \det\left[L_{n+i-3}^{(2)}\left(y\left(1+\frac{s}{x}\right)\right) \quad L_{n+i-1-j}^{(j)}\left(-x-s\right)\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}} \mathrm{d}y\right\}\mathrm{d}x
\end{aligned}$$

from which we obtain, after the variable transformation $y\left(1+s/x\right)=t$,

$$\begin{aligned}
\mathfrak{S}(s) &= (-1)^n \frac{(n-1)!}{\alpha!}\frac{e^{-\mu-sn}}{\mu^{n-1}}\\
&\quad\times \int_0^\infty \frac{e^{-xn}x^\alpha}{\left(1+\frac{s}{x}\right)^{(n-1)(n+\alpha)}} \det\left[\varrho_i(s,x) \quad L_{n+i-1-j}^{(j)}\left(-x-s\right)\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}} \mathrm{d}x
\end{aligned} \tag{C.9}$$

where

$$\varrho_i(s,x) = \int_0^\infty te^{-t} {}_0F_1\left(\alpha+1;\mu\left(x+\frac{t}{1+\frac{s}{x}}\right)\right) L_{n+i-3}^{(2)}(t)\mathrm{d}t \tag{C.10}$$

and we have used the fact that only the first column of the determinant depends on the variable t. The integral in (C.10) does not seem to have a simple closed-form solution. Therefore, to facilitate further analysis, we write the hypergeometric function with its equivalent power series expansion and use Lemma 3 to arrive at

$$\begin{aligned}
\varrho_i(s,x) &= \frac{1}{(n+i-3)!}\sum_{p=0}^{\infty}\sum_{k=0}^{p} \frac{\mu^p x^{p-k}(k+1)!}{k!(p-k)!(\alpha+1)_p}\frac{(1-k)_{n+i-3}}{\left(1+\frac{s}{x}\right)^k}\\
&= \frac{1}{(n+i-3)!}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \frac{\mu^{p+k} x^{p}(k+1)!}{k!p!(\alpha+1)_{p+k}}\frac{(1-k)_{n+i-3}}{\left(1+\frac{s}{x}\right)^k}.
\end{aligned}$$

The behavior of the Pochhammer symbol $(1-k)_{n+i-3}$ with respect to l deserves a special attention at this juncture. As such, we can observe that

$$(1-k)_{n+i-3} = \begin{cases} (n+i-3)! & \text{for } k=0 \\ 0 & \text{for } k=1 \\ (1-k)_{n+i-3} & \text{for } k \geq 2, \end{cases}$$

which enables us to decompose the terms corresponding to the summation index k into two parts. As a result, after some algebra, we obtain

$$\varrho_i(s,x) = {}_0F_1\left(\alpha+1;\mu x\right) + \frac{\sigma_i(s,x)}{(n+i-3)!}. \tag{C.11}$$

where

$$\sigma_i(s,x) = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \frac{\mu^{p+k+2}x^p(k+3)!}{(k+2)!p!(\alpha+1)_{p+k+2}} \frac{(-1-k)_{n+i-3}}{\left(1+\frac{s}{x}\right)^{k+2}}. \tag{C.12}$$

Now we substitute (C.11) into (C.9) and further simplify the resultant determinant using the multilinear property to obtain

$$\begin{aligned}\mathfrak{S}(s) = (-1)^n \frac{(n-1)!}{\alpha!}\frac{e^{-\mu-sn}}{\mu^{n-1}} \int_0^\infty e^{-xn}x^{n(n+\alpha-1)} \frac{{}_0F_1\left(\alpha+1;\mu x\right)}{(x+s)^{(n-1)(n+\alpha)}} \\ \times \det\left[1 \quad L_{n+i-1-j}^{(j)}\left(-x-s\right)\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}} \mathrm{d}x \\ + (-1)^n \frac{(n-1)!}{\alpha!}\frac{e^{-\mu-sn}}{\mu^{n-1}} \int_0^\infty \frac{e^{-xn}x^{\alpha}}{\left(1+\frac{s}{x}\right)^{(n-1)(n+\alpha)}} \\ \times \det\left[\frac{\sigma_i(s,x)}{(n+i-3)!} \quad L_{n+i-1-j}^{(j)}\left(-x-s\right)\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}} \mathrm{d}x.\end{aligned} \tag{C.13}$$

Let us now focus on further simplification of the determinant in the first integral. To this end, we apply the following row operation

$$i\text{th row} \to i\text{th row} + (-1)(i-1)\text{th row}$$

on each row for $i = 2, \ldots, \alpha+1$, and expand the resultant determinant using its first column to obtain

$$\det\left[1 \quad L_{n+i-1-j}^{(j)}\left(-x-s\right)\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}} = \det\left[L_{n+i-1-j}^{(j)}\left(-x-s\right)\right]_{i,j=1,\ldots,\alpha} \tag{C.14}$$

where we have used the contiguous relation given in (9). Therefore, in view of (C.7), we can clearly identify the first term in (C.13) as $-\mathfrak{P}(s)$. This key observation along with (C.3) gives

$$\begin{aligned}&\mathfrak{M}_V(s)\\&=(-1)^n\frac{(n-1)!}{\alpha!}\frac{e^{-\mu-sn}}{\mu^{n-1}}\\&\times\int_0^\infty\frac{e^{-xn}x^\alpha}{\left(1+\frac{s}{x}\right)^{(n-1)(n+\alpha)}}\det\left[\frac{\sigma_i(s,x)}{(n+i-3)!}\quad L^{(j)}_{n+i-1-j}\left(-x-s\right)\right]_{\substack{i=1,\ldots,\alpha+1\\j=2,\ldots,\alpha+1}}\mathrm{d}x.\end{aligned}$$

The remaining task at hand is to further simplify $\sigma_i(s,x)$ given in (C.12). To this end, following (10), we find that $(-1-k))_{n+i-3}$ is non-zero for $k\geq n+i-4$. Therefore, we shift the index k with some algebraic manipulation to obtain the m.g.f. of V as

$$\begin{aligned}&\mathfrak{M}_V(s)\\&=(n-1)!e^{-\mu-sn}\int_0^\infty\frac{e^{-xn}x^{n(n+\alpha)-1}}{(x+s)^{(n-1)(n+\alpha+1)}}\\&\qquad\times\det\left[\left(-\frac{\mu x}{x+s}\right)^{i-1}\vartheta_i(x\mu,x+s)\quad L^{(j)}_{n+i-1-j}\left(-x-s\right)\right]_{\substack{i=1,\ldots,\alpha+1\\j=2,\ldots,\alpha+1}}\mathrm{d}x\end{aligned}\tag{C.15}$$

where

$$\vartheta_i(w,z)=\frac{(n+i-1)}{(n+\alpha+i-2)!}\sum_{k=0}^{\infty}\frac{(n+i)_k(n+i-2)_k\,{}_0F_1\left(\alpha+n+i+k-1;w\right)w^k}{k!(\alpha+n+i-1)_k(n+i-1)_kz^k}.$$

Finally, we take the inverse Laplace transform of the above to yield the p.d.f. of V, which concludes the proof.

Appendix D. Proof of Theorem 9

We use partial fraction decomposition and exploit the symmetry to yield

$$\begin{aligned}\mathrm{E}_{\mathbf{\Lambda}}\left(\prod_{j=1}^n\frac{1}{z+\lambda_j}\right)=\frac{1}{(n-1)!}\int_{[0,\infty)^n}&\frac{1}{\prod_{j=2}^n(\lambda_j-\lambda_1)}\\&\times\frac{f_{\mathbf{\Lambda}}(\lambda_1,\ldots,\lambda_n)}{(z+\lambda_1)}\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n.\end{aligned}$$

To facilitates further analysis, we use (3) with some rearrangements to write

$$\mathrm{E}_{\mathbf{\Lambda}}\left(\prod_{j=1}^{n}\frac{1}{z+\lambda_j}\right)=\Omega_1(z)+\Omega_2(z) \tag{D.1}$$

where

$$\Omega_1(z)=\frac{(-1)^{n-1}}{\alpha!}\frac{e^{-\mu}}{\mu^{n-1}}\int_0^\infty\frac{{}_0F_1\left(\alpha+1,\mu\lambda_1\right)}{z+\lambda_1}\lambda_1^\alpha e^{-\lambda_1}\mathrm{d}\lambda_1 \tag{D.2}$$

and

$$\Omega_2(z)=\frac{\mathcal{K}_{n,\alpha}}{(n-1)!}\frac{e^{-\mu}}{\mu^{n-1}}\int_0^\infty\frac{\lambda_1^\alpha e^{-\lambda_1}}{z+\lambda_1}\left(\sum_{k=2}^{n}\int_{[0,\infty)^{n-1}}\frac{{}_0F_1\left(\alpha+1,\mu\lambda_k\right)}{\prod_{\substack{j=1\\j\neq k}}^{n}(\lambda_j-\lambda_k)}\right.$$
$$\left.\times\prod_{j=2}^{n}\frac{\lambda_j^\alpha e^{-\lambda_j}}{(\lambda_j-\lambda_1)}\Delta_n^2(\boldsymbol{\lambda})\mathrm{d}\lambda_2\cdots\mathrm{d}\lambda_n\right)\mathrm{d}\lambda_1. \tag{D.3}$$

Since further simplification of (D.2) seems an arduous task, we leave it in its current form and focus on (D.3). Noting that each term inside the summation contributes the same amount due to symmetry in $\lambda_2,\dots,\lambda_n$, we can further simplify (D.3) to yield

$$\begin{aligned}&\Omega_2(z)\\&=\frac{(-1)^n\mathcal{K}_{n,\alpha}}{(n-2)!}\frac{e^{-\mu}}{\mu^{n-1}}\int_0^\infty\frac{\lambda_1^\alpha e^{-\lambda_1}}{z+\lambda_1}\int_0^\infty{}_0F_1\left(\alpha+1,\mu\lambda_2\right)\lambda_2^\alpha e^{-\lambda_2}\\&\qquad\left(\int_{[0,\infty)^{n-1}}\frac{1}{\prod_{\substack{j=1\\j\neq 2}}^{n}(\lambda_j-\lambda_k)}\prod_{j=2}^{n}\frac{\lambda_j^\alpha e^{-\lambda_j}}{(\lambda_1-\lambda_j)}\Delta_n^2(\boldsymbol{\lambda})\mathrm{d}\lambda_3\cdots\mathrm{d}\lambda_n\right)\mathrm{d}\lambda_2\mathrm{d}\lambda_1.\end{aligned}$$

We now use the decomposition $\Delta_n^2(\boldsymbol{\lambda})=\prod_{j=2}^{n}(\lambda_1-\lambda_j)^2\prod_{j=3}^{n}(\lambda_2-\lambda_j)^2\Delta_{n-2}^2(\boldsymbol{\lambda})$ followed by the variable transformation $y_j=\lambda_{j-2},\ j=3,\dots,n$, to obtain

$$\begin{aligned}\Omega_2(z)=&\frac{(-1)^n\mathcal{K}_{n,\alpha}}{(n-2)!}\frac{e^{-\mu}}{\mu^{n-1}}\\&\times\int_0^\infty\frac{\lambda_1^\alpha e^{-\lambda_1}}{z+\lambda_1}\left(\int_0^\infty{}_0F_1\left(\alpha+1,\mu\lambda_2\right)\lambda_2^\alpha e^{-\lambda_2}U_{n-2}(\lambda_1,\lambda_2,\alpha)\mathrm{d}\lambda_2\right)\mathrm{d}\lambda_1\end{aligned}$$

where

$$U_n(r_1,r_2,\alpha):=\int_{[0,\infty)^n}\prod_{j=1}^{n}\prod_{i=1}^{2}(r_i-y_j)y_j^\alpha e^{-y_j}\Delta_n^2(\mathbf{y})\mathrm{d}y_1\cdots\mathrm{d}y_n. \tag{D.4}$$

The above integral can be solved using Mehta [47, Eqs. 22.4.2, 22.4.11] and Appendix E with the choice of $\mathsf{P}_k(x) = (-1)^k k! L_k^{(\alpha)}(x)$ to yield

$$U_n(r_1, r_2, \alpha) = (-1)n!(n+1)! \prod_{j=0}^{n-1} (j+1)!(j+\alpha)! \frac{\det \left[L_{n+i-1}^{(\alpha)}(r_j) \right]_{i,j=1,2}}{(r_2 - r_1)}. \tag{D.5}$$

This in turn gives

$$\begin{aligned} &\Omega_2(z) \\ &= (-1)^{n+1} \frac{(n-1)!}{\alpha!(n+\alpha-2)!} \frac{e^{-\mu}}{\mu^{n-1}} \\ &\times \int_0^\infty \frac{\lambda_1^\alpha e^{-\lambda_1}}{z+\lambda_1} \left(\int_0^\infty {}_0F_1\left(\alpha+1, \mu\lambda_2\right) \frac{\det \left[L_{n+i-1}^{(\alpha)}(\lambda_j) \right]_{i,j=1,2}}{(\lambda_2 - \lambda_1)} \lambda_2^\alpha e^{-\lambda_2} \mathrm{d}\lambda_2 \right) \mathrm{d}\lambda_1. \end{aligned} \tag{D.6}$$

Further manipulation of the above integral in its current form is highly undesirable due to the term $\lambda_2 - \lambda_1$ in the denominator. To circumvent this difficulty, we employ the following form of the Christoffel-Darboux formula (Abramowitz and Stegun [1, Eq. 22.12.1])

$$\frac{\det \left[L_{n+i-1}^{(\alpha)}(\lambda_j) \right]_{i,j=1,2}}{(\lambda_2 - \lambda_1)} = (-1) \frac{(n+\alpha-2)!}{(n-1)!} \sum_{j=0}^{n-2} \frac{j!}{(j+\alpha)!} L_j^{(\alpha)}(\lambda_1) L_j^{(\alpha)}(\lambda_2)$$

in (D.6) to obtain

$$\begin{aligned} \Omega_2(z) = \frac{(-1)^n}{\alpha!} \frac{e^{-\mu}}{\mu^{n-1}} \sum_{j=0}^{n-2} \frac{j!}{(j+\alpha)!} \int_0^\infty \frac{L_j^{(\alpha)}(\lambda_1)}{z+\lambda_1} \lambda_1^\alpha e^{-\lambda_1} \mathrm{d}\lambda_1 \\ \times \int_0^\infty {}_0F_1\left(\alpha+1, \mu\lambda_2\right) L_j^{(\alpha)}(\lambda_2) \lambda_2^\alpha e^{-\lambda_2} \mathrm{d}\lambda_2. \end{aligned}$$

Now the second integral can be solved using Lemma 3 to obtain

$$\int_0^\infty {}_0F_1\left(\alpha+1, \mu\lambda_2\right) L_j^{(\alpha)}(\lambda_2) \lambda_2^\alpha e^{-\lambda_2} \mathrm{d}\lambda_2 = \frac{\alpha!}{j!} (-\mu)^j e^\mu \tag{D.7}$$

which in turn gives

$$\Omega_2(z) = \frac{(-1)^n}{\alpha!} \frac{e^{-\mu}}{\mu^{n-1}} \sum_{j=0}^{n-2} \frac{(-\mu)^j \alpha!}{(j+\alpha)!} e^\mu \int_0^\infty \frac{L_j^{(\alpha)}(\lambda_1)}{z+\lambda_1} \lambda_1^\alpha e^{-\lambda_1} \mathrm{d}\lambda_1.$$

In order to further simplify the above integral, we rearrange the summation with respect to index j giving

$$\Omega_2(z) = \frac{(-1)^n}{\alpha!}\frac{e^{-\mu}}{\mu^{n-1}}\left(\sum_{j=0}^{\infty}\frac{(-\mu)^j\alpha!}{(j+\alpha)!}\, e^{\mu}\int_0^{\infty}\frac{L_j^{(\alpha)}(\lambda_1)}{z+\lambda_1}\lambda_1^{\alpha}e^{-\lambda_1}\mathrm{d}\lambda_1 - \sum_{j=n-1}^{\infty}\frac{(-\mu)^j\alpha!}{(j+\alpha)!}\, e^{\mu}\int_0^{\infty}\frac{L_j^{(\alpha)}(\lambda_1)}{z+\lambda_1}\lambda_1^{\alpha}e^{-\lambda_1}\mathrm{d}\lambda_1\right). \tag{D.8}$$

Let us now focus on the first infinite summation. As such, using (7) with some algebraic manipulation we obtain

$$\sum_{j=0}^{\infty}\frac{(-\mu)^j\alpha!}{(j+\alpha)!}L_j^{(\alpha)}(\lambda_1) = {}_0F_1(\alpha+1;\mu\lambda_1)e^{-\mu}.$$

Therefore, we can simpify (D.8), in view of (D.2), to yield

$$\Omega_2(z) = -\Omega_1(z) + (-1)^{n+1}\frac{1}{\mu^{n-1}}\sum_{j=n-1}^{\infty}\frac{(-\mu)^j}{(j+\alpha)!}\int_0^{\infty}\frac{L_j^{(\alpha)}(\lambda_1)}{z+\lambda_1}\lambda_1^{\alpha}e^{-\lambda_1}\mathrm{d}\lambda_1.$$

Finally, we shift the initial value of the summation index to zero and use Erdélyi [25, Eq. 6.15.2.16] with (D.1) to yield (26), which concludes the proof.

Appendix E. Proof of (A.4)

Following Mehta [47, Eqs. 22.4.2, 22.4.11], we begin with the integral

$$\int_{[0,\infty)^n}\prod_{j=1}^{n}e^{-y_j}\prod_{i=1}^{\alpha+1}(r_i-y_j)\Delta_n^2(\mathbf{y})\mathrm{d}y_1\cdots\mathrm{d}y_n = \prod_{i=0}^{n-1}(i+1)!i!\,\frac{\det\left[\mathsf{P}_{n+i-1}(r_j)\right]_{i,j=1,\ldots,\alpha+1}}{\Delta_{\alpha+1}(\mathbf{r})}, \tag{E.1}$$

where $\mathsf{P}_k(x)$'s are monic polynomials orthogonal with respect to e^{-x}, over $0\leq x<\infty$. As such, we choose $\mathsf{P}_k(x) = (-1)^k k! L_k^{(0)}(x)$, which upon substituting into the above equation gives

$$\int_{[0,\infty)^n}\prod_{j=1}^{n}e^{-y_j}\prod_{i=1}^{\alpha+1}(r_i-y_j)\Delta_n^2(\mathbf{y})\mathrm{d}y_1\cdots\mathrm{d}y_n = \tilde{K}\frac{\det\left[L_{n+i-1}^{(0)}(r_j)\right]_{i,j=1,\ldots,\alpha+1}}{\Delta_{\alpha+1}(\mathbf{r})}$$

where

$$\tilde{K} = (-1)^{(n-1)(\alpha+1)} \prod_{i=0}^{n-1} (i+1)!i! \prod_{i=1}^{\alpha+1} (-1)^i (n+i-1)!.$$

In general, the r_i's in the above formula are distinct parameters. However, for our purpose, we have to choose them in such a manner that the left side of (E.1) becomes $Q_n(a,b,\alpha)$. To this end, we select r_i's such that

$$r_i = \begin{cases} a & \text{if } i = 1 \\ b & \text{if } i = 2, \ldots, \alpha+1. \end{cases}$$

This direct substitution in turn gives 0/0 indeterminate form for the right side of (E.1). To circumvent this problem, instead of direct substitution, we follow an approach given in Khatri [43] to obtain

$$\begin{aligned} Q_n(a,b,\alpha) &= \tilde{K} \lim_{r_2,\ldots,r_{\alpha+1}\to b} \frac{\det\left[L^{(0}_{n+i-1}(a) \quad L^{(0)}_{n+i-1}(r_j)\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}}}{\det[a^{i-1} \quad r_j^{i-1}]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}}} \\ &= \tilde{K} \frac{\det\left[L^{(0)}_{n+i-1}(a) \quad \frac{\mathrm{d}^{j-2}}{\mathrm{d}b^{j-2}} L^{(0)}_{n+i-1}(b)\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}}}{\det\left[a^{i-1} \quad \frac{\mathrm{d}^{j-2}}{\mathrm{d}b^{j-2}} b^{i-1}\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}}}. \end{aligned} \tag{E.2}$$

The denominator of (E.2) gives

$$\det\left[a^{i-1} \quad \frac{\mathrm{d}^{j-2}}{\mathrm{d}b^{j-2}} b^{i-1}\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}} = \prod_{i=1}^{\alpha-1} i!\ (b-a)^{\alpha}. \tag{E.3}$$

The numerator can be simplified using (8) to yield

$$\begin{aligned} &\det\left[L^{(0)}_{n+i-1}(a) \quad \frac{\mathrm{d}^{j-2}}{\mathrm{d}b^{j-2}} L^{(0)}_{n+i-1}(b)\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}} \\ &\qquad = (-1)^{\frac{1}{2}\alpha(\alpha-1)} \det\left[L^{(0}_{n+i-1}(a) \quad L^{(j-2)}_{n+i+1-j}(b)\right]_{\substack{i=1,\ldots,\alpha+1\\ j=2,\ldots,\alpha+1}}. \end{aligned} \tag{E.4}$$

345 Substituting (E.3) and (E.4) into (E.2) gives (A.4).

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