

**Advanced Finite Element Formulations for Accurate and
Efficient Topology Optimization of 2D Continuum
Structures**

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Thesis/Dissertation submitted in partial fulfillment of the requirements for the degree
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DECLARATION

I declare that this is my own work and this thesis/dissertation does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any other University or Institute of higher learning, and to the best of my knowledge and belief, it does not contain any material previously published or written by another person except where the acknowledgement is made in the text. I retain the right to use this content in whole or in part in future works (such as articles or books).

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The candidate above has carried out research for the Master's thesis/dissertation under my supervision. I confirm that the declaration made above by the student is true and correct.

Name of Supervisor: H. M. S. T. Herath

Signature of the Supervisor:

Date: 05.12.2025

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ABSTRACT

Finite element analysis is crucial for topology optimization. Most implementations use 4-noded membrane elements (Q4) despite low accuracy from membrane locking. This highlights the need for alternatives balancing accuracy and computational cost. This work presents algorithms to implement advanced finite elements into topology optimization. The advanced elements include Pian-Sumihara (PS) and Allman elements, and quadratic membrane element (Q8). Three problem types are addressed: volume-constrained compliance minimization, stress-constrained volume minimization, and stress- and volume-constrained compliance minimization. Performance of different finite elements is analyzed and compared using multiple criteria: accuracy, convergence characteristics, and computational cost. For compliance minimization with uniform regular meshes, PS delivers the most accurate layouts. Its solutions closely resemble Mitchell's truss theory and follow load paths effectively. PS converges to the lowest compliance with smooth black-and-white designs. Q8 matches closely but yields slightly higher objective values. For distorted mesh problems, Q8 shows superior performance. For stress-constrained volume minimization, PS elements achieved optimal results with 0.226 volume fraction compared to Q4's 0.245. Allman and Q8 values are 0.241 and 0.237. Similar behavior occurred for volume- and stress-constrained compliance minimization. PS showed the most optimal design, while Q8 produced similar designs. Allman and Q4 failed to reach optimal designs, showing gray regions. Average computation times relative to Q4 vary: 1.06-1.19 $\times$ for PS, 1.39-1.74 $\times$ for Q8, and 1.55-2.54 $\times$ for Allman. Q4 is fastest but provides lower accuracy. With adequate h-refinement, Q4 can achieve results similar to advanced elements. PS elements are recommended for uniform regular meshes requiring high accuracy. Q8 elements are preferred for distorted mesh problems. Allman elements suit moderate accuracy requirements.

Keywords: Topology optimization, Finite element analysis, 2D continuum mechanics, Stress constraints, Compliance minimization

TABLE OF CONTENTS

Declaration	iii
Acknowledgement.....	iv
Abstract	v
Table of Contents	vi
List of Figures	ix
List of Tables.....	x
List of Abbreviations.....	xi
List of Appendices	xii
Chapter 1	1
Introduction	1
1.1 Background	1
1.2 Problem Statement	3
1.3 Significance of Research.....	4
1.4 Scope of the Study	4
1.5 Aim and Objectives.....	4
Chapter 2	1
LITERATURE REVIEW.....	1
2.1 Finite Element Analysis	1
2.2 Displacement-based Finite Elements Overview	2
2.3 Stiffness Matrix for Bilinear Quadrilateral Membrane (Q4) Finite Element.	4
2.4 Stiffness Matrix for Quadratic Quadrilateral Membrane (Q8) Finite Element..	7
2.5 Stiffness Matrix for Allman Finite Elements	9
2.6 Stiffness Matrix for Pian-Sumihara (PS) Finite Element.....	11
2.7 Global Matrix Assembly, Application of Boundary Conditions, and Solution of the Finite Element System	14
2.8 Rate of Convergence in H-Refinement using Strain Energy	14
2.9 Various Topology Optimization Problem Classes.....	15
2.9.1 Compliance function	16
2.9.2 Volume function.....	17

2.9.3 Stress function.....	17
2.9.4 Buckling function.....	19
Chapter 3	22
METHODOLOGY	22
3.1 Overview of Topology Optimization Framework.....	22
3.2 Compliance Minimization with Volume Constraints	22
3.3 Stress Constrained Volume Minimization Topology Optimization.....	25
3.4 Stress and Volume Constrained Compliance Minimization	29
Chapter 4	31
RESULTS AND DISCUSSION	31
4.1 Comparative Analysis of Different Finite Elements in FEA	31
4.1.1 Comparative Analysis of Finite Element in Uniform Regular Meshes	31
4.1.2 Comparative Analysis of Finite Elements in Distorted Meshes	36
4.1.3 Comparative Analysis of Finite Element in Problems with Varying Densities.....	39
4.2 Compliance Minimization with Volume Constrained Comparison.....	41
4.2.1 Mesh Dependency and Checkerboard Patterns.....	41
4.2.2 Computational cost across different finite elements in Compliance Minimization with Volume Constrained topology optimization	43
4.2.3 Accuracy and convergence capabilities across different finite elements in Compliance Minimization with Volume Constrained topology optimization...	44
4.2.4. Volume constrained compliance minimization problems with distorted mesh	47
4.3 Stress-Constrained Volume Minimization Topology Optimization Comparison 49	
4.4 Stress and Volume Constrained Compliance Minimization Topology Optimization Comparison	52
4.5 Ranking and Recommendations of Finite Elements for Topology Optimization 55	
4.6 Limitations and Research Prospects.....	56
Conclusions	58
References	60
Appendices.....	66

A1. Python-based Algorithm for Volume Constrained Compliance Minimization Topology Optimization Code using Q4 Elements	66
A2. Python based Algorithm for Volume Constrained Compliance Minimization Topology Optimization Code using Allman Elements	73
A3. Python based Algorithm for Volume Constrained Compliance Minimization Topology Optimization Code using PS Elements.....	74
A4. Python based Algorithm for Stress Constrained Volume Minimization Topology Optimization Code using Q8 Elements	75
A5. Python based Algorithm for Stress Constrained Volume Minimization Topology Optimization Code using Q4 Elements	76
A6. Python based Algorithm for Stress Constrained Volume Minimization Topology Optimization Code using Allman Elements	88

LIST OF FIGURES

Figure 1: Topology optimization problem.	2
Figure 2: Flow chart of topology optimization algorithms	2
Figure 3: Infinitesimal element in a 2D continuum.	2
Figure 4: Element in the reference (natural) and physical coordinate systems.....	5
Figure 5: Element in the reference and physical coordinate systems for Q8 elements	7
Figure 6: Allman element formulation.....	10
Figure 7: Geometry and boundary conditions of the cantilever beam benchmark problem.	31
Figure 8: SStress distributions for 20×8 mesh cantilever problem	35
Figure 9: Comparison of tip strain energy for the cantilever benchmark problem....	35
Figure 10: Geometry and boundary conditions of Cook's membrane problem.	37
Figure 11: Comparison of strain energy for Q4, Q8, Allman, and PS elements for the Cook's membrane benchmark problem	38
Figure 12: Stress distributions for the Cook's membrane benchmark problem.....	39
Figure 13: Stress distributions for the cantilever benchmark problem after 10th TO iteration with varying densities.	41
Figure 14: Mesh dependency analysis using the cantilever problem.....	42
Figure 15: Checkerboard pattern analysis using the cantilever benchmark problem.	43
Figure 16: Convergence curves for volume-constrained compliance minimization TO for benchmark problem.	45
Figure 17: PS element output resembles Mitchel's Truss theory	45
Figure 18: Geometry and boundary conditions of the L-beam benchmark problem.	46
Figure 19: Mesh dependency and checkerboard pattern analysis using the L beam benchmark problem.....	47
Figure 20: Setup of the annulus benchmark problem	48
Figure 21: Topology-optimized annulus designs.....	49
Figure 22: Optimized topologies and von Mises stress distributions.	51
Figure 23: Optimized topologies and von Mises stress distributions volume minimization, topology optimization.	54

LIST OF TABLES

Table 1: Gaussian quadrature parameters for numerical integration.	6
Table 2: Common TO problem classes	16
Table 3: Modification for stress-constrained topology optimization.....	18
Table 4: Comparison of tip displacement for the cantilever problem.....	32
Table 5: Normal stress comparison for the cantilever problem.	33
Table 6: Shear stress comparison for the cantilever problem.	34
Table 7: Computational cost comparison between element types	36
Table 8: Comparison of tip displacement for the Cook’s membrane benchmark problem.	37
Table 9: Comparison of tip displacement for the cantilever beam with TO-derived varying densities after.	39
Table 10: Computational time comparison in the volume constrained compliance minimization TO problem.....	44
Table 11: Computational time comparison in compliance minimization with volume constraint TO problems.....	51
Table 12: Computational time in compliance minimization with stress volume constraint TO problems.....	54
Table 13: Ranking of Q4, PS, Allman, and Q8 finite elements for 2D topology optimization	56

LIST OF ABBREVIATIONS

Abbreviation	Description
DOF	Degrees Of Freedom
FEA	Finite Element Analysis
FEM	Finite Element Method
KS	Kreisselmeier-Steinhauser
MMA	Method of Moving Asymptotes
OC	Optimality Criteria
PS	Pian-Sumihara element
Q4	4 noded bilinear quadrilateral membrane element
Q8	8 noded quadratic quadrilateral membrane element
SCTO	Stress Constrained Topology Optimization
SIMP	Solid Isotropic Material with Penalization
TO	Topology Optimization

LIST OF APPENDICES

<u>A1. Python based Algorithm for Volume Constrained Compliance Minimization Topology Optimization Code using Q4 Elements</u>	60
<u>A2. Python based Algorithm for Volume Constrained Compliance Minimization Topology Optimization Code using Allman Elements</u>	67
<u>A3. Python based Algorithm for Volume Constrained Compliance Minimization Topology Optimization Code using PS Elements</u>	68
<u>A4. Python based Algorithm for Stress Constrained Volume Minimization Topology Optimization Code using Q8 Elements</u>	69
<u>A5. Python based Algorithm for Stress Constrained Volume Minimization Topology Optimization Code using Q4 Elements</u>	70
<u>A6. Python based Algorithm for Stress Constrained Volume Minimization Topology Optimization Code using Allman Elements</u>	82